Topology Study Guide

Josh Levenberg

November 22, 1999

The Fundamental Group

If X is path connected, $a, b \in X$ connected by a path γ, then there is an isomorphism from $\pi_1(X, a)$ to $\pi_1(X, b)$ given by $[\alpha] \to [\overline{\gamma}\alpha\gamma]$. Consequently "$\pi_1(X)$" is well defined.

Glueing Lemma *Given f is defined by $f(x) = \begin{cases} g(x) & x \in A \\ h(x) & x \in B \end{cases}$ where: g and h are continuous on those subspaces, g and h agree on $A \cap B$, and A, B are either both closed or both open. Then f is continuous.*

Definition Two maps $f_0, f_1 : X \to Y$ are *homotopic* if there exists a continuous map $F : X \times I \to Y$ such that $F|X \times \{0\} = f_0$ and $F|X \times \{1\} = f_1$. The maps are *homotopic rel A* if this map fixes $a \times I$ for all $a \in A$.

Definition Two spaces are *homotopy equivalent*, written $X \simeq Y$, if there exists continuous maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$. Note this is an equivalence relation.

Covering Spaces

Definition A *covering space* is a continuous map $p : E \to X$ such that each point x_0 in X has a nbhd U with $p^{-1}(U) = \coprod_{i \in I} U_i$ where $p|U_i$ is a homeomorphism for all i. E is called the *total space*, X is the *base space*, $p^{-1}(x)$ is the *fiber at x*, and U is an *evenly-covered neighborhood.* Note that fibers are discrete, p is surjective, and a local homeomorphism. Also X has the quotient topology, with p the quotient map.

Uniqueness of lifts *Assume Y is a connected space. If $\tilde{f}, \tilde{\tilde{f}} : Y \to E$ are two lifts of a fixed map $f : Y \to X$ which agree at some point of Y, then $\tilde{f} = \tilde{\tilde{f}}$.*

Proof Define $A \subseteq Y$ to be the set of points where $\tilde{f}, \tilde{\tilde{f}}$ agree. We know A nonempty by assumption, want to show A open and closed. Assume $a \in A$. Choose an evenly covered nbhd of $f(a)$, $U = \cup_i U_i$. Let M be the component of $f^{-1}(U)$ containing a. $\tilde{f}(M), \tilde{\tilde{f}}(M)$ are connected sets in E which intersect at at least one point (since $a \in A$). Since connected, each is contained in a single U_i. We know $p|U_i \circ \tilde{f}|M = p|U_i \circ \tilde{\tilde{f}}|M$ apply $(p|U_i)^{-1}$ (a homeomorphism) to both sides to get $\tilde{f}|M = \tilde{\tilde{f}}|M$. Therefore A is open. Applying the same reasoning to $b \notin A$ gives $\tilde{f}(M) \subseteq U_i$ and $\tilde{\tilde{f}}(M) \subseteq U_j$, $i \neq j$. Therefore $\tilde{f}, \tilde{\tilde{f}}$ disagree on all of M, so A's complement is open. $\surd$

Path-Lifting Lemma *Let $\gamma : I \to X$ be a path starting at x_0. Fix $\tilde{x_0} \in p^{-1}(x_0)$. Then γ has a unique lift $\tilde{\gamma} : I \to E$ starting at $\tilde{x_0}$.*

Proof Cover the image of γ with evenly covered neighborhoods. By compactness of $\gamma(I)$ there is a finite subcover, so we may disect I into finitely many closed intervals, each of which is contained in one evenly covered neighborhood. Now lift γ one interval at a time. Start with $\tilde{\gamma}(0) = \tilde{x_0}$. The lift of each successive interval starts at the endpoint of the lift of the last. On each interval, the lift is computed using $(p|U_i)^{-1}$, which is uniquely defined by the starting point. By glueing lemma, the result is continuous. $\surd$

Lebesque Covering Lemma *If X a compact metric space and $\{U_i\}_{i \in I}$ is an open cover of X, then there exists a $\delta > 0$ such that any set with diameter $< \delta$ is contained in a U_i for some i (though i may not be unique).*

Homotopy Lifting Lemma *Assume $f : Y \to X$ has a lift $\tilde{f} : Y \to E$ and that $F : Y \times I \to X$ is a homotopy with $F|Y \times \{0\} = f$. Then F has a unique lift $\tilde{F} : Y \times I \to E$ with $\tilde{F}|Y \times \{0\} = \tilde{f}$.*

Proof For any point $y \in Y$, we may cover $F(\{y\} \times I)$ with finitely many evenly covered neighborhoods by compactness. Intersect their pre-images to get open $V_y \subseteq Y$, with $y \in V_y$. The image $F(V_y \times I)$ is covered by finitely many evenly covered neighborhoods U_{yi}. As before compute $\tilde{F}$ on $V_y \times I$. It will agree with $\tilde{F}$ on $V_z \times I$ on the intersection by uniqueness, so $\tilde{F}$ is well defined on all of Y. Finally it will be continuous by the glueing lemma. $\surd$

Corollary *There does not exist a continuous retraction from the disc to its boundry, $S^1 \xrightarrow{i} D \xrightarrow{r} S^1$ with $r \circ i = \mathrm{id}_{S^1}$.*

Proof Otherwise, it would induce maps of their fundamental groups. However since D is contractible, $\pi_1(D) = 0$, so the identity map on $\pi_1(S^1) = \mathbf{Z}$ can not factor through $\pi_1(D)$. $\surd$

Brower Fixed Point Theorem (dim 2) *Any continuous map f of the disc to itself has a fixed point.*

Proof Otherwise we would have a map $r(x) = \frac{x - f(x)}{|x - f(x)|}$ contradicting the previous corollary. $\surd$

Definition An action of a group G on a space E

Definition An action of G on E is *properly discontinuous* if every point $x \in E$ has a nbhd U such that all translates of U are disjoint. Note this implies that there are no fixed points under the action of notrivial elements of G.

Definition If G acts properly discontinuously E then we may form the quotient $E \to E/G$ defined by $e \mapsto [Ge]$, and giving the result the quotient topology.

Theorem *If G acts properly discontinuously on a path-connected simply-connected space E, then $\pi_1(E/G) \cong G$.*

Proof Let $X = E/G$. Pick a base point $e_0 \in E$, then let $x_0 = [Ge_0]$ be the base point for X. We would like to define $\psi : G \to \pi_1(E/G)$. Choose a path γ_g from e_0 to $g(e_0)$ for each $g \in G$. Observe $q \circ \gamma_g$ is a loop based at x_0 in X. Let $\psi(g) = [q \circ \gamma_g] \in \pi_1(X, x_0)$. We need to show this map is well-defined, homomorphic, injective, and surjective.

Its well-defined since given γ'_g a different path, $\gamma_g \simeq \gamma'_g$ since E is simply-connnected. Therefore $[q \circ \gamma_g] = [q \circ \gamma'_g]$.

The map is a homomorphism since given $g, h \in G$, let T_g represents the action of g on E, let $\gamma \cdot \alpha$ represent path concatenation, then $\psi(gh) = [q \circ \gamma_{gh}] = [q \circ (\gamma_g \dot{(} T_g \gamma_h))] =$

$[q\circ(\gamma_g)][q\circ(T_g\gamma_h))] = [q\circ\gamma_g][q\circ\gamma_h] = \psi(g)\psi(h)$.

The map is surjective, since given any loop α at x_0 lifts to a path $\tilde{\alpha}$ starting at e_0, and ending at $g(e_0)$ for some $g \in G$. Therefore $\psi(g) = [\alpha]$.

Finally, the map is injective since given an element $g \in G$ such that $\psi(g) = 1$, we must have that there exists a path γ_g from e_0 to $g(e_0)$ which projects to a null-homotopic loop rel $\{0,1\}$. So there is a homotopy F between the constant loop at x_0 and $q\circ\gamma_g$. By lifting this homotopy, we get $\tilde{F}$ which is a homotopy between the constant loop at e_0 and the path γ_g rel $\{0,1\}$. Clearly g is forced to be the identity in G. $\surd$

Example Consider the unit quaternions, a group equivalent to $SU(2)$ topologically equal to S^3 (contained in the quaternions or $\mathbf{R}^4$). $SU(3)$ acts on quaternions by conjugation $h \mapsto q^{-1}hq$. It fixes the real axis, preserves magnitude, and therefore rotates rigidly the space spanned by i, j, k. This gives a map $SU(2) \to SO(3)$ which is actually a homomorphism. However it takes antipodes to the same point. Since $SU(2) \equiv S^3$ is simply-connected and path-connected the theorem gives $\pi_1(SO(3)) \cong \mathbf{Z}_2$.

Let I be *binary icosehedral group*, the group of rotational symmetries of a regular dodecahedron in $\mathbf{R}^3$, $I \subseteq SO(3)$. Gives diagram:

$$\begin{array}{cc} I \leq & SU(2) \\ \uparrow & \downarrow \\ A_5 \leq & SO(3) \end{array}$$

Now I acts by right multiplication on $SU(2)$ (or equivalently S^3). The quotient $SU(2)/I$ is a 3-manifold with $\pi_1 \cong I$ (not abelian, not S_5, also not $\mathbf{Z}_2 \times A_5$), a *Poincaré homology 3-sphere.*

Lemma *Fibers $p^{-1}(x)$ have the same cardinality over each path-component of X.*

Proof Pick $x, y \in X$ and a path γ between them. Lifting the path gives a map $p^{-1}(x) \to p^{-1}(y)$. This map is invertible (lift γ^{-1}), so this map is a bijection. $\surd$

Lemma 2 *$p_* : \pi_1(E, e_0) \to \pi_1(X, x_0)$ is injective (where $e_0 \in p^{-1}(x_0)$). We therefore can think of $\pi_1(E, e_0)$ as a subgroup of $\pi_1(X, x_0)$.*

Proof As before, given a loop α in E such that $p_*[\alpha] = 1$, there is a homotopy F of $p_*[\alpha]$ with the constant loop at x_0, rel $\{0,1\}$. Lifting this homotopy gives a homotopy between α and the constant loop at e_0. $\surd$

Lemma 3 *There exists a monodromy homomorphism $\theta : \pi_1(X, x_0) \to Aut(p^{-1}(x_0))$ (permutations of the fiber over x_0), defined by lifting a loop α to a path $\tilde{\alpha}$ starting at $e \in p^{-1}(x_0)$ and setting $\theta([\alpha])(e) = \tilde{\alpha}(1)$.*

Proof The map is well-defined since a homotopic path rel $\{0,1\}$ lifts to a path with the same end point (and in fact is still homotopic rel $\{0,1\}$). So for every element of $\pi_1(X, x_0)$ we get a map $(p^{-1}(x_0) \to p^{-1}(x_0))$. Composition of paths maps to composition of maps by uniqueness. These maps are permutations since they have inverses given by $\theta([\alpha]^{-1})$. $\surd$

Lemma 4 *Given $[\alpha] \in \pi_1(X, x_0)$, then $\theta([\alpha])(e) = e \iff [\alpha] \in p_*\pi_1(E, e)$.*

Proof ($\Longrightarrow$) If α lifts to a path $\tilde{\alpha}$ starting at e which happens to end at e as well, then $[\alpha] = [p\circ\tilde{\alpha}] = p_*[\tilde{\alpha}]$ since $\tilde{\alpha}$ is indeed a loop, and therefore $[\tilde{\alpha}]$ is a member of $\pi_1(E, e)$.

($\Longleftarrow$) If $[\alpha] = p_*[\beta]$ for some $[\beta] \in \pi_1(E, e)$, then α lifts to a path starting at e that is homotopic rel $\{0,1\}$ to β, and therefore ends at e. $\surd$

Lemma 5 *If E is path-connected and $e \in p^{-1}(x_0)$ then there is a bijetion*

$$p_*\pi_1(E,e)\backslash\pi_1(X,x_0) \leftrightarrow p^{-1}(x_0).$$

Proof Define the map $\pi_1(X,x_0) \to p^{-1}(x_0)$ by $[\alpha] \mapsto \theta([\alpha])(e)$. This map is surjective by path-connectedness, ie, given any $e' \in p^{-1}(x_0)$, there is a path γ from e to e', and $\theta([p\circ\gamma])(e) = e'$. Suppose $[\alpha], [\beta] \in \pi_1(X,x_0)$ map to the same e', we want that $[\alpha][\beta]^{-1}$ lies in $p_*\pi_1(E,e)$, or equivalently (see Lemma 4) $\theta([\alpha][\beta]^{-1})(e) = e$. This last equality clearly holds since the lifts of $[\alpha], [\beta]$ starting at e both end at e'. Therefore $p_*\pi_1(E,e_0)[\alpha] = p_*\pi_1(E,e_0)[\beta]$ as cosets. $\surd$

Definition A space Y is said to be *locally-path-connected* if every neighborhood of a point has a path-connected sub neighborhood.

Map lifting *Let* Y be path-connected and locally-path-connected. Let $f : (Y,y_0) \to (X,x_0)$ be a continuous map, and let $(E,e_0) \xrightarrow{p} (X,x_0)$ be a covering space. Then f has a lift $\tilde{f} : Y \to E$ with $\tilde{f}(y_0) = e_0 \iff f_*\pi_1(Y,y_0) \le p_*\pi_1(E,e_0)$.

Proof ($\Longrightarrow$) If there is a lift, then $p\tilde{f} = f$ and so the image of f_* factors through p_*.

($\Longleftarrow$) Given $f_*\pi_1(Y,y_0) \le p_*\pi_1(E,e_0)$, and a $y \in Y$, want to define $\tilde{f}(y)$. Since Y is path connected, there is a path γ from y_0 to y. Lift $f\gamma$ starting at e_0 and take $(\widetilde{f\gamma})(1)$ as the definition of $\tilde{f}(y)$. If γ' is a different path from y_0 to y, represent γ' by the homotopic $(\gamma' \cdot \gamma^{-1}) \cdot \gamma$. Since homotopies lift, this substitution won't change the endpoint of the lift. $f \circ \widetilde{((\gamma' \cdot \gamma^{-1}) \cdot \gamma)}$ is equal to $f \circ \widetilde{(\gamma' \cdot \gamma^{-1})} \cdot \widetilde{(f \circ \gamma)}$. Since $\gamma' \cdot \gamma^{-1}$ is a loop, it is in $\pi_1(Y,y_0)$, and $[f \circ (\gamma' \circ \gamma^{-1})] \in f_*\pi_1(Y,y_0) \le p_*\pi_1(E,e_0)$. By lemma 4, the lift of $f \circ (\gamma' \circ \gamma^{-1})$ starting at e_0 ends at e_0. We conclude $(\widetilde{f\gamma'})(1) = (\widetilde{f\gamma})(1)$ and $\tilde{f}$ is well defined.

Aside: we have shown that $f_*\pi_1(Y,y_0) \le p_*\pi_1(E,e_0)$ implies that any loop α with $\alpha(0) = \alpha(1) = y_0$ lifts to a loop $\tilde{\alpha}$ with $\tilde{\alpha}(0) = \tilde{\alpha}(1) = e_0$.

To show $\tilde{f}$ is continuous, consider a point $y \in Y$. The point $f(y)$ has an evenly covered neighborhood U_y. Let V_y be a path-connected neighborhood of y contained in $f^{-1}(U_y)$. By the above argument, we may define $\tilde{f}$ on V_y only considering paths which begin at y_0, go through y, and then remain within V_y. Since $f(V_y) \subseteq U_y$, an evenly covered neighborhood, the lift of f is contained in some U_i covering U_y. In fact, on V_y, $\tilde{f} = (p|U_i)^{-1} \circ f$ which is the composition of continuous maps. $\surd$

Corollary *If $E \to X$ is a covering space, then $\pi_n(E) \cong \pi_n(X)$ for all $n \ge 2$.*

Proof Any map f of a sphere to X lifts to a map to E, since spheres (of dim $n \ge 2$) are simply-connected and therefore satisfy $f_*(\{1\}) \le p_*\pi_1(E)$. By the above theorem, therefore $\pi_n(E)$ surjects onto $\pi_n(X)$. This is injective as well since homotopies between maps of a sphere into X lift. $\surd$

Definition Covering spaces E, E' over X are *equivalent* or *isomorphic* if there exists a *fiber-preserving* homeomorphism $\phi : E \to E'$ where

$$\begin{tikzcd} E \arrow[r, "\phi"] \arrow[dr, "p"'] & E' \arrow[d, "p'"] \\ & X \end{tikzcd}$$

commutes. A *homomorphism* of covers is any map (not necessarily a homeomorphism) which preserves. Isomorphisms of E to itself are called *automorphisms*, *covering transformations*, or *deck translations*. Note that the deck translations of a cover form a group.

Note Assume all covering spaces are path-connected and locally-path-connected from now on.

Proposition *(1) E, e_0 and E', e_0' on X are equivalent by a homeomorphism ϕ taking e_0 to e_0' iff $p_*\pi_1(E, e_0)$=$p_*'\pi_1(E', e_0')$ as subgroups of $\pi_1(X, x_0)$.*

(2) E, e_0 and E' are equivalent (without any control on base points) iff $p_*\pi_1(E, e_0)$ and $p_*\pi_1(E', e_0')$ are conjugate subgroups of $\pi_1(X, x_0)$.

Proof 1) ($\Longrightarrow$) Obvious, since $\phi p' = p$ and $\phi_*(\pi_1(E, e_0)) = \pi_1(E', e_0')$.

($\Longleftarrow$) Since $p_*(\pi_1(E, e_0)) \leq p_*'(\pi_1(E', e_0'))$, we may lift p to a map $\phi : E \to E'$, and similarly lift p' to get ψ. Note that $\phi(e_0) = e_0'$, $\psi(e_0') = e_0$, and they are unique. Then $\phi \circ \psi$ is a lift of p' with respect to cover E' over X which sends $e_0' \mapsto e_0'$, and is therefore the identity by uniqueness. Similarly in reverse $\psi \circ \phi$ is the identity on E.

2) ($\Longrightarrow$) If E, e_0 and E', e_0' are equivalent by a homeomorphism ϕ, then ϕ sends e_0 to e_0'', some point in the fiber over x_0. Consequently $\phi_*(\pi_1(E, e_0)) = \pi_1(E', e_0'')$. Choose a path γ from e_0'' to e_0' (using path-connectedness). Then $\pi_1(E', e_0'') \cong \pi_1(E', e_0')$ by sending $[\alpha] \mapsto [\gamma^{-1}\alpha\gamma]$, ie. $\pi_1(E', e_0'') = \gamma_\# \pi_1(E', e_0')$. Now apply $p_*'\phi_*\pi_1(E, e_0) = p_*\pi_1(E, e_0)$. Observe $p_0'(\gamma^{-1}\alpha\gamma) = (p' \circ \gamma)^{-1}(p' \circ \alpha)(p' \circ \gamma)$ since $p' \circ \gamma$ is a loop. Finally, $p_*\pi_1(E, e_0) = p_*'\phi_*\pi_1(E, e_0) = [p' \circ \gamma]^{-1} p_*'\pi_1(E', e_0')[p' \circ \gamma]$, which shows they are conjugate.

($\Longleftarrow$) Same argument in reverse: start with $[\alpha]^{-1}\pi_1(E', e_0')[\alpha] = \pi_1(E, e_0)$, write $[\alpha]$ as $[p \circ \gamma]$ for γ a lift of α. $\surd$

Classification of deck translations theorem *Let $(E, e_0) \xrightarrow{p} (X, x_0)$ (each path-connected and locally-path-connected), $G = \pi_1(X, x_0)$, and $H = p_*\pi_1(E, e_0) \leq G$. Let K be the group of covering transformations*

$$\begin{array}{ccc} E & \xrightarrow{\phi} & E \\ & \searrow & \downarrow \\ & & X \end{array}$$

Then (1) given points $e_1, e_2 \in p^{-1}(x_0)$, there exists a covering transformation which sends e_1 to e_2 iff $p_*\pi_1(E, e_1) = p_*\pi_1(E, e_2)$. And this covering transformation is unique.

Proof By a lifting argument. The lifts $\phi : (E, e_1) \to (E, e_2)$ and $\psi : (E, e_2) \to (E, e_1)$ exist if and only if the group conditions are satisfied. Uniqueness comes from the uniqueness of lifts since E is path-connected. $\surd$

(2) *K acts on E properly discontinuously, so $E \to E/K$ and $E/K \to X$ are covers.*

Proof Fix $e \in E$, take an evenly-covered neighborhood of $p(e)$ and let U_i be the sheet containing e. Claim: for $g_1, g_2 \in K$, $g_1U_i \cap g_2U_i \neq \emptyset \Longrightarrow g_1 = g_2$. If nonempty, there exists e_1, e_2 in U_i s.t. $g_1e_1 = g_2e_2$. Applying p gives, $p(g_1e_1) = p(e_1) = p(e_2) \Longrightarrow e_1 = e_2$ (since they are in the same fiber, in the same sheet). Thus $g_1e_1 = g_2e_1$, and $(g_2^{-1}g_1)e_1 = e_1 \Longrightarrow g_2^{-1}g_1$ is the identity by uniqueness. Therefore $g_1 = g_2$. $\surd$

(3) *$E/K \cong X \iff K$ acts transitively on the fiber, ie. given $e_1, e_2 \in p^{-1}(x_0)$, there exists a $g \in K$ taking $g(e_1) = e_2$. If true, $E \to X$ is called a* regular *or* normal *cover.*

(4) *H is a normal subgroup of $G \iff E \to X$ is a regular/normal cover.*

Proof ($\Longrightarrow$) Pick e_0, e_1 in the fiber $p^{-1}(x_0)$. There exists a deck translation sending $e_0 \mapsto e_1 \iff p_*\pi_1(E, e_0) = p_*\pi_1(E, e_1)$. By definition $p_*\pi_1(E, e_0) = H$, which we know to be normal in G, so its conjugates are all H. Further $p_*\pi_1(E, e_1)$ is a conjugate of H by $[p \circ \gamma]$ for γ a path connecting e_0 to e_1.

($\Longleftarrow$) Suppose $[\alpha] \in \pi_1(X, x_0)$. Lift α to a path γ based at e_0. Then $[\alpha]^{-1}H[\alpha] = [p \circ \gamma]^{-1}p_*\pi_1(E, e_0)[p \circ \gamma] = p_*\pi_1(E, \gamma(1))$. Since there exists a deck translation taking e to $\gamma(1)$ (by regularity), $p_*\pi_1(E, \gamma(1)) = H$ (by 1). $\surd$

(5) *The group K is isomorphic to $\frac{\text{Normalizer}(H \text{ in } G)}{H} = \{g \in G \text{ s.t. } g^{-1}Hg = H\}$ (the biggest subgroup of G in which H is normal) for any cover. For a regular cover $K \cong G/H$.*

Proof Define $\psi : N(H) \to K$ by lifting $[\alpha] \in N(H)$ to γ based at e_0, and setting $\psi([\alpha])$ to the deck translation that sends e_0 to $\gamma(1)$. This exists since by the above, $p_*\pi_1(E, \gamma(1)) = p_*\pi_1(E, e_0)^{[\alpha]}$ (that means conjugation by $[\alpha]$) $= p_*\pi_1(E, e_0)$ (since $[\alpha] \in N(H)$). ψ is a homomorphism since $\psi([\alpha][\beta]) = \psi([\alpha])\psi([\beta])$, both are the deck translation that sends e_0 to the end point of the lift of $\alpha \cdot \beta$ based at e_0 (since the lift of β starting at the end point of the lift of α is the image of the lift based at e_0 after the translation $\psi([\alpha])$, by uniqueness, so "projection commutes with composition"). Finally, the kernel consists of $[\alpha] \in N(H)$ which lift to loops in $\pi_1(E, e_0)$, ie. $p_*\pi_1(E, e_0) \cap N(H) = H$ since $p_*\pi_1(E, e_0) = H \leq N(H)$. $\surd$

Definition A loop is *null-homotopic* if it is homotopic rel $\{0, 1\}$ to a constant loop. If a loop is not null-homotopic, it is said to be *essential.*

Definition A space X is *semi-locally simply-connected* (*slsc*) if every point $x \in X$ has a neighborhood U such that every loop in U (based at x) is null-homotopic in X. Or equivalently $\pi_1(U, x) \to \pi_1(X, x)$ has trivial image.

Definition A cover $\tilde{X} \to X$ (with $X, \tilde{X}$ both path-connected) is called a *universal cover* ('the universal cover' once we show it is unique) if $\pi_1(\tilde{X})$ is trivial, ie. $\tilde{X}$ is simply-connected.

Remark *If X has a universal cover $\tilde{X}$, then X is slsc.*

Proof Given x, take an evenly-covered nbhd U containing x. Then any loop in U lifts to a loop in $\tilde{X}$ (since $p|\tilde{U_i}$ is a homeomorphism). Since $\tilde{X}$ is simply-connected, the lifted loop is null-homotopic in $\tilde{X}$. Projecting this homotopy down using p shows the original loop is null-homotopic as well. $\surd$

Theorem *If X is slsc, locally path-connected, connected, and path-connected, then X has a universal cover $\tilde{X}$.*

Proof 1) Define PX to be path space $\{\gamma : I \to X | \gamma(0) = x_0\}$. Define $\tilde{X}$ as the quotient of PX obtained by identifying $\alpha \sim \beta \iff \alpha(1) = \beta(1)$ and $\alpha \simeq \beta$ rel $\{0, 1\}$. Write $\langle\alpha\rangle$ for the equivalence class containting α.

2) X has a base of open sets consisting of neighborhoods of points such that (by slsc) $\pi_1 U \xrightarrow{i_*} \pi_1 X$ is trivial, and U is path-connected (by lpc). Define basic open sets for $\tilde{X}$ to be $\langle\alpha, U\rangle = \{\langle\alpha \cdot \sigma | \ \sigma$ is a path from $\alpha(1)$ contained inside U $\}$. This is a base since the intersection of two such is a union of other basic open sets. Suppose $\langle\alpha, U\rangle \cap \langle\beta, V\rangle \neq \emptyset$, so there exists a $\langle\gamma\rangle$ in the intersection. Want that $\langle\gamma, U \cap V\rangle$ is a basic open set. Suppose $\langle\delta\rangle \in \langle\gamma, U \cap V\rangle$. There exists a ρ contained in $U \cap V$ going from $\gamma(1)$ to $\delta(1)$ such that $\delta \simeq \gamma \cdot \rho$ rel $\{0, 1\}$. And there exists a path σ such that $\gamma \simeq \alpha \cdot \sigma$ rel $\{0, 1\}$. Therefore $\delta \simeq \alpha \cdot \sigma \cdot \rho$ and $\langle\delta\rangle \in \langle\alpha, U\rangle$. Similarly $\langle\delta\rangle \in \langle\beta, V\rangle$.

3) Define projection $p : \tilde{X} \to X$ by $\langle\alpha\rangle \mapsto \alpha(1)$ (well-defined). Sends a basic open set $\langle\alpha, U\rangle$ to U (since U is path-connected). Also for basic open U in X, $p^{-1}(U) = \bigcup_{\alpha|\alpha(1)\in U}\langle\alpha, U\rangle$. This shows p is both open and continuous.

4) To show p is a local homeomorphism, observe that p is a bijection when restricted to $\langle\alpha, U\rangle \to U$ (by the slsc property that $\pi_1 U \to \pi_1 X$ is trivial).

5) Need to check that a basic open set U is evenly-covered. Want that if $\langle\alpha, U\rangle \cup \langle\beta, U\rangle \neq \emptyset$

then they're equal. Pick $\langle\gamma\rangle$ in the intersection. Suppose $\langle\delta\rangle \in \langle\alpha, U\rangle$, then there exists a ρ s.t. $\delta \simeq \alpha\rho$ rel $\{0,1\}$, also there exists σ, τ s.t. $\gamma \simeq \alpha \cdot \sigma$ rel $\{0,1\}$ and $\gamma \simeq \beta \cdot \tau$ rel $\{0,1\}$. Therefore $\delta \simeq \beta \cdot \tau \cdot \sigma^{-1}\rho$ rel $\{0,1\}$ and $\langle\alpha, U\rangle \subseteq \langle\beta, U\rangle$. Argument is symmetric so $\langle\beta, U\rangle \subseteq \langle\alpha, U\rangle$.

So $\tilde{X} \xrightarrow{p} X$ is a covering space, since $p^{-1}(U) = \bigcup\langle\alpha, U\rangle$.

6) $\tilde{X}$ is path-connected. Choose a basepoint $\tilde{x_0} = \langle e_{x_0}\rangle \in \tilde{X}$. Any point $\langle\gamma\rangle \in \tilde{X}$ can be joined to $\langle e_{x_0}\rangle$ by the path $I \to \tilde{X}$ defined by $s \mapsto \langle t \mapsto \gamma(st)\rangle$.

Lets compute $p_*\pi_1(\tilde{X}, \tilde{x_0})$: a loop α in X based at x_0 is in this subgroup of $\pi_1(X, x_0)$ $\iff$ its lift starting at $\tilde{x_0}$ is a loop. In fact α lifts to the path in $\tilde{X}$, $\tilde{\alpha}(s) = \langle t \mapsto \alpha(st)\rangle$ $I \to \tilde{X}$. Observe $\tilde{\alpha}(0) = \tilde{x_0}$ and $(p \circ \tilde{\alpha})(s) = \alpha(s)$ so $\tilde{\alpha}$ is a lift of α starting at $\tilde{x_0}$, therefore by uniqueness it is the lift. $\tilde{\alpha}(1) = \langle\alpha\rangle$ so if α lifts to a loop then $\langle\alpha\rangle = \tilde{x_0} = \langle e_{x_0}\rangle \Longrightarrow \alpha(1) = x_0$ and α is null-homotopic $\Longrightarrow [\alpha] = 1 \in \pi_1(X, x_0)$. So $p_*\pi_1(\tilde{X}, \tilde{x_0}) = \{1\} \leq \pi_1(X, x_0)$ but p_* is injective so $\tilde{X}$ is simply-connected. $\surd$

Theorem $\pi_1(X, x_0)$ *acts properly discontinuously on* $\tilde{X}$ *with* $G\backslash\tilde{X} \cong X$. *Further, any subgroup* $H \leq G$ *also acts p.d. with quotient* $H\backslash\tilde{X} \stackrel{\text{def}}{=} E$. *This* E *is a cover of* X *with* $p_*\pi_1(E) = H$.

Moral Covering spaces corresponding to every subgroup of $\pi_1(X, x_0)$ exist.

Proof Given $[\alpha] \in \pi_1(X, x_0)$ and $\langle\gamma\rangle \in \tilde{X}$, define $[\alpha]\langle\gamma\rangle = \langle\alpha\cdot\rangle$ (well-defined). Since $[\alpha]\langle\gamma, U\rangle = \langle\alpha \cdot \gamma, U\rangle$, the action of $[\alpha]$ is open, continuous, and bijective ($[\alpha]^{-1}$ gives the inverse action). It is a homeomorphism and satisfies the algebraic requirements of an action: if $\theta([\alpha])$ is the homeomorphism associated with $[\alpha]$, then $\theta([\alpha])\theta([\beta]) = \theta([\alpha][\beta])$. If $[\alpha]\langle\gamma, U\rangle \cap [\beta]\langle\gamma, U\rangle \neq \emptyset \Longrightarrow$ they are equal by stage 5 of previous theorem $\alpha \cdot \gamma \simeq \beta \cdot \gamma$ rel $\{0,1\}$ (since they end at the same point) $\Longrightarrow \alpha \simeq \beta$ rel $\{0,1\}$. So the action is properly discontinuous. So $G\backslash\tilde{X}$ has fundamental groooup G and is $\cong X$ (check that G is acting transitively on fibers at x_0.) To prove the last part, take as the base point of $H\backslash\tilde{X}$ the coset $H\tilde{x_0}$. $p_*\pi_1(H\backslash\tilde{X}, H\tilde{x_0})$ consists of loops in X, x_0 which lift to paths in $\tilde{X}$ starting at $\tilde{x_0}$ and ending at a translate $h\tilde{x_0}$ ($h \in H$) $\Longrightarrow p_*\pi_1(E, H\tilde{x_0}) = H$. $\surd$

Digression: Free Groups

Definition Given a set A of *letters*, we can define a group $F(A)$, the *free group on* A, by (1) define a *word* to be a finite string of symbols belonging to $A \cup A^{-1} \cup 1$ (eg. if $A = \{a, b\}$, words are strings of $\{a, b, a^{-1}, b^{-1}, 1\}$). Define the equivalence relation on words by $aa^{-1} \mapsto 1$, $a^{-1}a \mapsto 1$, $x1y \mapsto xy$. Define multiplication on $F(A)$ by juxtaposition of words. 1 is the identity, and the inverse of $abab^{-1}a^{-1}aaa$ is $a^{-1}a^{-1}a^{-1}aba^{-1}b^{-1}a^{-1}$. Eg. $F(\{a\}) = \mathbf{Z}$ but $F(\{a,b\}) \neq \mathbf{Z} \times \mathbf{Z}$ since its not abelian. The universal property of $F(A)$ is that it is the group such that any group G and function $A \to G$ extends to a unique homeomorphism $F(A) \to G$.

Definition *Generators and Relations*: Let A be a set, called the *Generators*, and R be a subset of $F(A)$, called the relators. They can produce a group $G = \langle A : R\rangle$ defined as $F(A)$ quotient by the minimal normal subgroup of $F(A)$ containing the set R, ie. the subgroup generated by all conjugates (by elements of $F(A)$) of all elements of R. If G is a group $\cong \langle A : R\rangle$ for some A, R, this is called a *presentation* of G. Example: $\langle a, b : a^2 = b^3 = (ab)^5 = 1\rangle$ ('relations' rather than 'relators') $\cong$ double cover of A_5 (120 elements). $\langle a, b : a^2 = b^3 = (ab)^7 = 1\rangle$ is infinite.

Definition If G, H are two groups, their *free product*, $G * H$, is defined by taking $F((G - \{1\}) \cup (H - \{1\}))$ and imposing the relations from G and from H separately. Ex. $\cdots xg_1g_2y \cdots \mapsto \cdots x(g_1g_2)y \cdots$ where (g_1g_2) is the element of G corresponding to the product of g_1 and g_2. Universal property: If $\phi : G \to K$, $\psi : H \to K$ are homomorphisms then $G * H$ has the property that there exists a homomorphism $\phi * \psi : G * H \to K$, $\langle a\rangle * \langle b\rangle = \langle a, b\rangle$.

Definition Given three groups G, H, N and injections $i : N \hookrightarrow G$, $j : N \hookrightarrow H$, can form *free product with amalgamation* $G *_N H = (G * H)$ quotient by the normal subgroup generated by $\{i(n)j(n)^{-1} : \forall n \in N\}$. Universal property: $G *_N H$ is the group (well-defined up to isomorphism) containing G, H as subgroups and such that given K, $\theta : G \to K$, $\phi : H \to K$ with "θ and ϕ agreeing on N" ($\theta \circ i = \phi \circ j$) then there exists a unique extension to a homomorphism $G *_N H \to K$.

Van Kampen's Theorem *Let X, x_0 be path-connected and docomposed as a union $X = A \cup B$, with A, B open path-connected sets containing x_0 and $A \cap B$ also path-connected. Then $\pi_1(X, x_0) = \pi_1(A, x_0) *_{\pi_1(A\cap B, x_0)} \pi_1(B, x_0)$ (maps used are i_*, j_* where $i : A \cap B \hookrightarrow A$ and $j : A \cap B \hookrightarrow B$ are inclusions).*

Remark The definition of $G *_{N,i,j} H$ works fine even if i, j aren't inclusions.

Example $\pi_1(S^{n\geq 2}, x) = \{1\}$ for x a point on the equator.

Proof Decompose S^n into $(S^n - \{N\}) \cup (S^n - \{S\})$ (both parts are homeomorphic to R^n by stereographic projection). The intersection is path-connected for $n \geq 2$ (and is in fact $\simeq S^{n-1}$, but this is irrelevant) so $\pi_1(S^{n\geq 2}) = 1 *_{\pi_1(S^{n-1})} 1 = 1$ fails if $n = 1$ since the intersection is disconnected. $\surd$

Proof (Outline) Consider a map $\pi_1(A) * \pi_1(B) \to \pi_1(X)$. An element of the LHS is a word of [loops] in A and B. Compose as paths and regard result as in $\pi_1(X)$ — a homomorphism. To show surjective, use the Lebesque Covering Lemma: a loop γ leads to a covering of I by $\gamma^{-1}(A)$ and $\gamma^{-1}(B)$. Disect into length δ intervals and cut the path at each of these points, either the paths continues in the same open set or if it switches, chop and connect γ(this point) to x_0 inside $A \cap B$ (uses the path-connectedness of $A \cap B$). In this way replace γ by a homotopic loop which is a composite of loops in A or in $B \implies$ surjective. Kernel? Again subdivide to get a homotopy in X between (a composite of loops in A and B) and (the constant loop at x_0). Subdivide the homotopy into squares contained in $F^{-1}(A), F^{-1}(B)$. Only interested in things homotopic to the identity loop... $\surd$

CW-complexes

Definition A *CW-complex* is formed inductively by gluing cells. Start with X^0, the *0-skeleton*, which is a disjoint union of points $\coprod_{\alpha\in I^{(0)}} P_\alpha$, the set of points/vertices. Next add X^1 the *1-skeleton* which is $X^0 \sqcup \coprod_{\alpha\in I^{(1)}} I_\alpha$ quotient by identifications. These identifications are given by an attaching map for each I_α, $\phi_\alpha : \partial I_\alpha \to X^0$. Identification is, $x \in \partial I_\alpha \sim \phi_\alpha(x) \in X^0 \subseteq X^0 \sqcup \coprod_{\alpha\in I^{(1)}} I_\alpha$. Similarly the *2-skeleton* is a collection of B^2 which are attached along their boundry to the 1-skeleton. Repeat infinitely. Topology at each stage is given by the quotient. Define the *characteristic map* for a cell B^n_α as $\Phi_\alpha : B^n_\alpha \hookrightarrow X^{n-1} \coprod_{\alpha\in I^{(n)}} B^n_\alpha \to X^n$ via the nth quotient map. Observe that $\Phi_\alpha | \partial B^n_\alpha = \phi_\alpha$. By the definition of quotient topology, U is open in $X^n \iff \Phi_\alpha^{-1}(U)$ is open in $B^n_\alpha \forall\alpha$ and $i^{-1}_{n-1}(U)$ open in X^{n-1}, where

$i_{n-1} : X^{n-1} \hookrightarrow X^n$. By induction, U is open $\iff$ $\Phi_\alpha^{-1}(U)$ is open in $B_\alpha^n \forall n, \alpha$. Similarly for U closed.

TO DO find an intrinsic definition of CW complex

Example Join ∞ many unit intervals at a point. Consider points $\{\frac{1}{n}\} \in I_n^1$. These do NOT converse to "0!"

Remark Complexes are Hausdorff, locally contractible (therefore lpc, slsc).

Definition A *graph* is a 1-dim CW-complex with no cells of dim ≥ 2. Assume also no disjoint points. Use Van Kampen's Theorem to calculate $\pi_1(G)$ for G a connected graph with finitely many 1-cells. First take a maximal subtree T (of a path-connected graph), a sub-cell complex which is path-connected and simply-connected. Second, recall homotopy of attaching maps does not affect *homotopy-type* of a space (question 13 on an assignment). Since a maximal tree will contain all vertices, and is path-connected, homotop so all 1-cells in $G - T$ attach at one point. Thirdly, T is contractible to this point, so G is homotopic to a "boquet of circles" (a connected graph with one vertex). Then write the bouquet of circles as the union of open sets, each set U_i has one point missing from the interior of all but one edge. $\pi_1(U_i) = \mathbf{Z}$ and the intersection of all the U_i is contractible. By Van Kampen's Theorem, π_1 of the bouquet is the free product of $\mathbf{Z}$s, one for every circle, ie. the free group with n generators.

Remark We may extend this technique for 2-complexes, etc. We know $\pi_1(\text{graph})$ is freely generated by "loops." A 2-complex is a graph union some 2-cells (attached by maps $S^1 \to$graph). The homotopy of the resulting complex depends only on $\phi_*(C) \in \pi_1(X^1)$.

Example Given a bouquet of two circles a, b, and an attached 2-cell c, $\pi_1 \cong \langle a, b : \phi_*(c) = 1\rangle \cong \langle a, b\rangle$ quotient by the smallest normal subgroup containing $\phi_*(c)$.

Corollary *Any group with a finite presentation is* π_1*(some space).*

Proof Take a bouquet: one loop for each generator. Attach one disc for each relation. $\surd$

Remark *Any cover of a CW-complex is also a CW-complex.*

Proof Since B_α is simply-connected, we may lift the characteristic maps of cells in the base $\Phi_\alpha : B_\alpha \to X$ to $\Phi_a \tilde{l}pha : B_\alpha \to E$. $\surd$

Corollary *Any subgroup of a free group is free.*

Proof Recall that a subgroup defines a covering space. Since the base space is a bouquet of circles this covering space is again a graph. By the previous argument its fundamental group (which is isomorphic to the subgroup we are interested in) is free. $\surd$

Covers of ∞ The universal cover is the "TV aerial," a 1-complex (graph) vertices $\longleftrightarrow$ elements of $\langle a, b\rangle$. Edge run between vertices of the form (w, wa), (w, wb), (w, wa^{-1}), (w, wb^{-1}). Locally can be drawn in $\mathbf{R}^2$, but the metric is not homogeneous. There is an action of $\langle a, b\rangle$ on the left, ie. given $w \in \langle a, b\rangle$, get a map taking '1'$\mapsto$'$w$' and takes the rest of the graph accordingly. To construct a cover with subgroup $\langle a\rangle \leq \langle a, b\rangle$, take quotient by action of subgroup. Get a loop plus aerial off of b and b^{-1}. Similarly for $\langle a^2\rangle$ get two points connected with two edges, plus four aerials. Recall, $\frac{N(\langle a\rangle)}{\langle a\rangle} \cong$ the group of dec translations of $\langle a\rangle$.

Simplicial Homology

Definition A set of points $(a_0, \ldots, a_n)$ in $\mathbf{R}^n$ is *affinely independent* if $\sum_{i=0}^n \lambda_i a_i = 0$ and $\sum_{i=0}^n \lambda_i = 0$ implies all $\lambda_i = 0$. Equivalently a_i are affinely independent if $\{a_1 - a_0, a_2 - a_0, \ldots, a_n - a_0\}$ are linearly independent. Affinely independent points $\{a_0, \ldots, a_n\}$ span an *n-simplex* also written $\sigma = (a_0, \ldots, a_n) = \{\sum_{i=0} \lambda_i a_i : \sum \lambda_i = 1, 0 \leq \lambda_i \leq 1\}$. $\sum \lambda_i a_i$ is the barycenter (center of gravity) of weights λ_i at each point a_i. These λ_i $(i = 0 \ldots n)$ are *barycentric coordinates* of points inside a simplex (easy to check uniqueness).

Remarks A simplex is the convex hull of $\{a_0, \ldots, a_n\}$. a_i are its vertices. A simplex is closed, bounded (compact) subspace in $\mathbf{R}^N$. It has an *interior* $\sigma^\circ = \{\sum \lambda_i a_i : \lambda_i \neq 0 \forall i\}$, and a *boundary* $\partial\sigma = \sigma - \sigma^\circ$. Any subset of $\{a_0, \ldots, a_n\}$ defines a face τ of σ, by taking the convex hull of those points. Write $\tau \leq \sigma$ to mean "is a face of", or $\tau < \sigma$ to indicate τ is a *proper face* not equal to σ. σ is the dijoint union of the interiors of its faces (including σ itself). Note that the interior of a vertex is defined to be itself.

Definition A *simplicial complex* K is a set of simplexes in $\mathbf{R}^N$ satisfying (i) $\forall \sigma \in K, \tau < \sigma \Longrightarrow \tau \in K$, and (ii) $\sigma, \tau \in K \Longrightarrow \sigma \cap \tau = \emptyset$ or it is a single face of both σ and τ.

Remark An *abstract simplicial complex* is just a collection K of subsets of some set V of "vertices" satisfying the axioms. Can find a geometric realisation of a finite abstract complex by identifying th vertex set with basis vectors $\{e_0, e_1, \ldots, e_{|V|}\}$ in $\mathbf{R}^{|V|}$.

Definition A *subcomplex* is a subset which is also a complex, written $L \leq K$. The *n-skeleton*, $K^{(n)} = \{$ all simplexes of dim$\leq n\} \subseteq K$. The *polyhedron*, $|K| = \bigcup_{\sigma \in K} \sigma \subseteq \mathbf{R}^N$ is a topological space (subset of $\mathbf{R}^N$). If K is finite, then $|K|$ is compact Hausdorff.

Definition An *oriented* simplex is $\sigma = (a_0, \ldots, a_n)$ with the vertices ordered and considered modulo even permutations. There exist two orientations for every σ.

Definition An *n-chain* on a complex K is a $\mathbf{Z}$-linear combination of oriented n-simplexes in K modulo the relation $1(\sigma) \sim (-1)\sigma^{\text{opp}}$, where σ^{opp} is σ with the opposite orientation.

Definition The *n-th chain group* $C_n(K)$ is the abelian group of chains under pointwise addition.

Remark $C_n(K) \cong \bigoplus^i \mathbf{Z}$ where i is the number of n-simplexes in K.

Definition The *boundary map* $\partial : C_n K \to C_{n-1} K$ is given by the linear extension of $\partial(a_0, \ldots, a_n) = \sum_{i=0}^n (-1)^i (a_0 \cdots \hat{a}_i \cdots a_n)$ (the hat indicates that all but a_i are present) where $(a_0, \ldots, a_n)$ is an oriented n-simplex. Check that $\partial(a_{\pi(0)}, \ldots, a_{\pi(n)}) = (\text{sgn}\pi)\partial(a_0, \ldots, a_n)$ when π is a permutation of $\{0 \ldots n\}$, so ∂ is well defined.

Lemma *The composition $C_n K \xrightarrow{\partial_n} C_{n-1} K \xrightarrow{\partial_{n-1}} C_{n-2} K$, "$\partial_n^2$"$\equiv \partial_{n-1} \circ \partial_n$ is zero for all n.*

Remark Define $C_n(K) = 0$ if $n < 0$ or if K has no cells of dim n.

Proof

$$\begin{aligned}
&\partial\left(\sum_{i=0}^n (-1)^i (a_0 \cdots \hat{a}_i \cdots a_n)\right) \\
&= \sum_{i=0}^n (-1)^i \left[\sum_{j<i} (-1)^j (a_0 \cdots \hat{a}_j \cdots \hat{a}_i \cdots a_n) + \sum_{j>i} (-1)^{j-1} (a_0 \cdots \hat{a}_i \cdots \hat{a}_j \cdots a_n)\right] \\
&= \sum_{i,j=0, j<i}^n (-1)^{j+i} (a_0 \cdots \hat{a}_j \cdots \hat{a}_i \cdots a_n) + \sum_{i,j=0, j<i}^n (-1)^{j+i-1} (a_0 \cdots \hat{a}_j \cdots \hat{a}_i \cdots a_n)
\end{aligned}$$

$$= \quad 0$$

$\surd$

Remark This shows we have an example of a chain complex.

Definition A *chain complex* is a sequence $0 \to \cdots \to C_n \overset{\partial_n}{\to} C_{n-1} \overset{\partial_{n-1}}{\to} C_{n-2} \to \cdots \to C_1 \to 0$ of abelian groups and homomorphisms ∂ such that "$\partial^2 = 0$" (really $\partial_{n-1} \circ \partial_n = 0 \forall n$).

Definition The *n-cycles*, $Z_n = \ker(\partial_n) \le C_n$ and the *n-boundary*, $B_n = \operatorname{im}(\partial_{n+1}) \le C_n$. Define the n^{th} *homology* of a chain complex to be $H_n = Z_n/B_n$ the quotient group.

Remark If $Z_n = B_n \forall n$, the sequence is called *exact* $\implies H_n = 0 \forall n$.

Example For a single point you get the chain complex $0 \to C_0 \cong \mathbf{Z} \to 0$, so $H_0(\text{pt}) \cong \mathbf{Z}$, all other trivial.

Example For the boundary of a triangle with vertices a,b,c, have the chain complex $0 \to C_1 \cong \mathbf{Z}^3 \to C_0 \cong \mathbf{Z}^3 \to 0$ with boundary maps $\partial(ab) = b - a$, $\partial(bc) = c - b$, and $\partial(ca) = a - c$. Therefore $Z_0 = \langle a, b, c\rangle$, $Z_1 = \ker \partial = \{n(ab+bc+ca) : n \in \mathbf{Z}\} = \langle ab+bc+ca\rangle$, $B_0 = \operatorname{im} \partial = \langle b - a, c - b, a - c\rangle$ (remark: $C_1/Z_1 \cong B_0 \cong \mathbf{Z}^2$), $B_1 = 0$. Therefore $H_0 \cong \mathbf{Z}$ with the generator (any of the three vertices)$+B_0$ (a coset), and $H_1 \cong \mathbf{Z}$ with generator $(ab + bc + ca) + B_1$. Observe that $B_n \le Z_n$ so the quotient makes sense. Every boundary is a cycle since $\partial(B_n) = \partial(\partial C_{n+1}) = 0$ since $\partial^2 = 0$.

Definition A *simplicial map* $f : |K| \to |L|$ is a function/map taking simplexes in K to simplexes in L affinely. Determined by the motion of the vertices. In other words, f is determined by its restriction to a map from the vertex set of K to the vertex set of L by $f\left(\sum_{i=0}^n \lambda_i a_i\right) = \sum_{i=0}^n \lambda_i f(a_i)$ where λ_i are barycentric coordinates. Note that f is not necessarily injective or surjective, but respects the simplicial complex structure beyond the polyhedron structure. Also the image of an n-simplex may be a simlex of smaller dimension (like a triangle being sent to a line segment, two vertices being mapped into a single vetex of the line segment).

Definition A simplicial map f induces maps $\{f_n\} = f_* : C_nK \to C_nL$ (homomorphism) by

$$f_n(a_0, \ldots, a_n) = \begin{cases} (f(a_0), \ldots, f(a_n)) & \text{if this is an } n\text{-simplex without repeats} \\ 0 & \text{otherwise} \end{cases}$$

Lemma *(1) f_* forms a* chain map *"$\partial \circ f_* = f_* \circ \partial$" where the diagram*

$$\begin{array}{ccc} \to C_nK & \xrightarrow{\partial K} & C_{n-1}K \to \\ \downarrow f_n & & \downarrow f_{n-1} \\ \to C_nL & \xrightarrow{\partial L} & C_{n-1}L \to \end{array}$$

commutes. (2) $f_*(Z_nK) \le Z_nL$ and $f_*(B_nK) \le B_nL$. (3) f_* induces a homomorphism $H_nK \to H_nL$ (also called f_*). (4) If f, g are simplicial maps $|K| \overset{f}{\to} |L| \overset{g}{\to} |M|$, then $(g \circ f)_* = g_* \circ f_*$ and $(_K)_* = \{\mathrm{id}_{H_n(K)}\}$.

Proof (1) $\partial f_*(a_0, \ldots, a_n) = f_*(\partial(a_0, \ldots, a_n)) = \sum_{i=0}^n (-1)^i \left(f(a_0) \cdots \widehat{f(a_i)} \cdots f(a_n)\right)$ if there are no repeats amongst $(f(a_0), \ldots, f(a_n))$. If at least 3 of the $f(a_i)$ are equal (or two pairs), then $\partial f_* = f_*\partial = 0$. Otherwise if exactly two are equal, WLOG $f(a_0) = f(a_1)$, $\partial f_*(a_0, \ldots, a_n) = 0$, and

$$f_*\partial(a_0, \ldots, a_n) = f_*\left((\hat{a}_0 a_1 \cdots a_n) - (a_0 \hat{a}_1 \cdots a_n) + \sum_{i=2}^n (-1)^i (a_0 a_1 \cdots \hat{a}_i \cdots a_n\right)$$

in which the first two terms cancel, and the remaining terms are 0. $\surd$

(2) Let $z \in Z_n K$. $\partial(f_* z) = f_* \partial z = f_*(0) = 0 \Longrightarrow f_* z$ is a cycle. Similarly for B_n.

(3) $H_n K = Z_n K / B_n K$ so $f_*(z + B_n K) = f_*(z) + B_n L$ gives a well-defined induced map, by (2).

(4) Easy check.

Moral $H_* = \{H_n\}$ is a functor from (simplicial complexes with simplicial maps) to (collections of abelian groups and collections of homomorphisms). C_* does not have the same nice properties with respect to the connecting homomorphisms.

Definition Let f_*, g_* be two chain maps between chain complexes $C_* K$ and $C_* L$. If there exists a sequence of maps $h_n : C_n K \to C_{n+1} L$ satisfying $\partial h + h \partial = f_* - g_*$ (in all dimensions, really the equation is $\partial_{n+1,L} \circ h_n + h_{n-1} \circ \partial_{n,K} = f_n - g_n$), then f_* and g_* are *chain homotopic*.

$$\begin{array}{ccccccc}
 & \to C_{n+1}K & \xrightarrow{\partial_{n+1,K}} & C_n K & \xrightarrow{\partial_{n,K}} & C_{n-1}K & \\
h_{n+1} \swarrow & & h_n \swarrow & f_n \downarrow\downarrow g_n & \swarrow h_{n-1} & & \\
C_{n+2,L} \to & C_{n+1}L & \xrightarrow{\partial_{n+1,L}} & C_n L & \xrightarrow{\partial_{n,L}} & C_{n-1}L &
\end{array}$$

Lemma *Chain-homotopic chain maps induce the same homomorphism on homology.*

Proof

$$\begin{aligned}
(f_n - g_n)(z + B_n K) &= (\partial h + h \partial)(z + B_n K) \\
&= \partial_L h(z + B_n K) + h(0) \\
&\subseteq \partial_L(C_{n+1} L) \\
&= B_{n+1} L = 0 \in H_n L
\end{aligned}$$

$\surd$

Definition If K is a complex, and a is a vertex in $\mathbf{R}^N$ affinely independent from K, then the *cone on* K $aK = K \cup \{a\sigma : \sigma \in K\} \cup \{a\}$ where if $\sigma = (a_0, \ldots, a_n)$ is an n-simplex, then $a\sigma = (a, a_0, \ldots, a_n)$ is an $n+1$-simplex. Alternatively, could add $e_{N+1} \in \mathbf{R}^N \times \mathbf{R}$. It is easy to check that aK is a simplicial complex.

Theorem *If aK is a cone on K, then* $H_*(aK) \cong H_*(a) \cong \begin{cases} \mathbf{Z} & \textit{dim 0} \\ 0 & \textit{otherwise} \end{cases}$

Corollary *If Δ^n is the n-simplex union all its faces,*

$$H_*(\Delta^n) = H_*(\mathrm{pt}) = \begin{cases} \mathbf{Z} & \textit{dim 0} \\ 0 & \textit{otherwise} \end{cases}$$

Proof (Corollary) $(a_0, \ldots, a_n) = a_0(a_1, \ldots, a_n)$. Induct. $\surd$

Proof (Theorem) Define $i : |a| \hookrightarrow |aK|$ to be inclusion, and $j : |aK| \to a$ to be smashing to a point. $j \circ i$ is $\mathrm{id}_{|a|}$ so $j_* \circ i_*$ is an isomorphism. Want to show $i \circ j : |aK| \to |aK|$ is chain-homotopic to the identity on $|aK|$ ($\Longrightarrow i_*$ and j_* are isomorphisms $H_*(aK) \overset{\Longrightarrow}{\longleftarrow} H_*(a)$). Define $h_n : C_n(aK) \to C_{n+1}(aK)$ by $h(a) = 0$, $h(\sigma^n) = a\sigma^n$, $h(a\sigma^n) = 0$. Now check the formula. Note $\partial(a\sigma) = \sigma - a \cdot \partial\sigma$ (where $\cdot$ is coning operation extended linearly to a homomorphism) meaning

$$\partial(a, a_0, \ldots, a_n) = (a_0, \ldots, a_n) - \sum_{i=0}^{n} (-1)^i (a, a_0, \ldots, \hat{a}_i, \ldots, a_n)$$

Compare $(\partial h + h\partial)$ to $f_* - g_*$ in all possible cases (for $v \neq a$ and $n \geq 2$):

$$\begin{aligned}
(\partial h + h\partial)(a) &= \partial(0) + h(0) = 0 = \mathrm{id}_*(a) - (i \circ j)_*(a) = f_*(a) - g_*(a) \\
(\partial h + h\partial)(v) &= \partial(av) + h(0) = v - a = \mathrm{id}_*(v) - (i \circ j)_*(v) = f_*(v) - g_*(v) \\
(\partial h + h\partial)(\sigma^n) &= \partial(a\sigma) + h(\partial(\sigma)) = \sigma - a \cdot \partial\sigma + a \cdot \partial\sigma = \sigma \\
&= \mathrm{id}_*(\sigma) - (i \circ j)_*(\sigma) = f_*(\sigma) - g_*(\sigma) \\
(\partial h + h\partial)(a\sigma^n) &= \partial(0) + h(\sigma - a \cdot \delta\sigma) = h(\sigma) = a\sigma \\
&= \mathrm{id}_*(a\sigma) - (i \circ j)_*(a\sigma) = f_*(a\sigma) - g_*(a\sigma)
\end{aligned}$$

$\surd$

Theorem (1) $H_i(\Delta^n) = \begin{cases} \mathbf{Z} & i = 0 \\ 0 & otherwise \end{cases}$

(2) $H_i(\partial\Delta^n) = \begin{cases} \mathbf{Z} & i = 0, n-1 \\ 0 & otherwise \end{cases}$

Remark $|\Delta^n| \cong B^n$ and $|\partial\Delta^n| \cong S^{n-1}$.

Proof (1) Done. (2) Consider inclusion map $\partial\Delta^n \overset{i}{\hookrightarrow} \Delta^n$. Since they have the same complex except in dim n, we get the diagram:

$$\begin{array}{ccccccc}
0 \to & 0 & \to C_{n-1}\partial\Delta^n & \xrightarrow{\partial} & C_{n-2}\partial\Delta^n & \to \cdots \\
& \downarrow & \|\downarrow & & \| & \\
0 \to & C_n\Delta^n \to & C_{n-1}\Delta^n & \xrightarrow{\partial} & C_{n-2}\Delta^n & \to \cdots
\end{array}$$

We know $Z_i(\partial\Delta^n) = Z_i(\Delta^n)\forall i \leq n-1$, $B_i(\partial\Delta^n) = B_i(\Delta^n)\forall i \leq n-2$, so $H_i(\partial\Delta^n) \cong H_i(\Delta^n)\forall i \leq n-2$. Since there are no simplexes with dim$\geq n$, $H_i(\partial\Delta^n) = 0\forall i \geq n$. Which leaves $H_{n-1}(\partial\Delta^n) = Z_{n-1}(\partial\Delta^n)/B_{n-1} = Z_{n-1}(\Delta^n)$ since $Z_{n-1}(\partial\Delta^n) = Z_{n-1}(\Delta^n)$ and $B_{n-1} = 0$. Since $H_{n-1}(\Delta^n) = Z_{n-1}/B_{n-1} = 0$, $Z_{n-1}(\Delta^n) = B_{n-1}(\Delta^n) = C_n(\Delta^n) \cong \mathbf{Z}$. So $H_{n-1}(\partial\Delta^n) \cong \mathbf{Z}$ and is generated by the boundary of the missing n-simplex. $\surd$

Exact Sequences

Definition A sequence of abelian groups and homomorphisms $\to A_n \xrightarrow{\alpha_n} A_{n-1} \xrightarrow{\alpha_{n-1}} A_{n-2} \to \cdots$ is *exact* if $\ker(\alpha_{n-1}) = \mathrm{im}\,(\alpha_n)\forall n$. Note: $\mathrm{im}\,\alpha_n \leq \ker\alpha_n \implies A_*$ is a chain complex, adding $\alpha_n \geq \ker\alpha_n \implies H_*$(chain complex) is 0.

Definition A *short exact sequence* (short=3) is a sequence $0 \to A \to B \to C \to 0$ which is exact. So s.e.s. is just $A \xrightarrow{\alpha} B \xrightarrow{\beta} C$ with α injective, β surjective, and $B/\alpha(A) \overset{\beta}{\cong} C$.

Example $0 \to \mathbf{Z} \xrightarrow{\times 2} \mathbf{Z} \xrightarrow{\mathrm{mod}\ 2} \mathbf{Z}_2 \to 0$.

Remark $0 \to A \to B \to 0$ exact $\implies A \simeq B$ and $0 \to A \to 0$ exact $\implies A \simeq 0$.

Definition A *s.e.s. of chain complexes* is $0 \to A_* \xrightarrow{\alpha_*} B_* \xrightarrow{\beta_*} C_* \to 0$ meaning A_*, B_*, C_* are chain complexes and α_*, β_* are chain maps with $0 \to A_n \to B_n \to C_n \to 0$ exact $\forall n$. Therefore all squares in the diagram:

$$\begin{array}{ccccccc}
& \partial\downarrow & & \partial\downarrow & & \partial\downarrow & \\
0 \to & A_n & \xrightarrow{\alpha_n} & B_n & \xrightarrow{\beta_n} & C_n & \to 0 \\
& \partial\downarrow & & \partial\downarrow & & \partial\downarrow & \\
0 \to & A_{n-1} & \xrightarrow{\alpha_{n-1}} & B_{n-1} & \xrightarrow{\beta_{n-1}} & C_{n-1} & \to 0 \\
& \partial\downarrow & & \partial\downarrow & & \partial\downarrow &
\end{array}$$

commute, all the δ maps are different, and vertical chains are not necessarily exact.

Fundamental theorem of homological algebra *If* $0 \to D_* \overset{\alpha_*}{\to} E_* \overset{\beta_*}{\to} F_* \to 0$ *is a s.e.s. then there is a* long exact sequence *(long* $\Rightarrow > 3$*) of homology groups*

$$\to H_n(D_*) \overset{\alpha_*}{\to} H_n(E_*) \overset{\beta_*}{\to} H_n(F_*) \overset{\partial_*}{\to} H_{n-1}(D_*) \to \cdots \overset{\partial_*}{\to} H_{n-2}(D_*) \to \cdots$$

Proof The piece $H_nD_* \overset{\alpha_*}{\to} H_nE_* \overset{\beta_*}{\to} H_nF_*$ exists just by the properties of chain maps.

$$\begin{array}{ccccccccc}
 & & & & u \in E_{n+1} & & v \in F_{n+1} & & \\
 & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \\
0 & \to & t \in D_n & \xrightarrow{\alpha_n} & x \in E_n & \xrightarrow{\beta_n} & z \in F_n & \to & 0 \\
 & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \\
0 & \to & w \in D_{n-1} & \xrightarrow{\alpha_{n-1}} & y \in E_{n-1} & \xrightarrow{\beta_{n-1}} & F_{n-1} & \to & 0 \\
 & & \partial \downarrow & & \partial \downarrow & & \partial \downarrow & & \\
 & & D_{n-2} & \xrightarrow{\alpha_{n-2}} & E_{n-2} & & & &
\end{array}$$

To define ∂_*, start with a homology class $[z] \in H_n(F_*)$, where z is a cycle in $Z_n(F_*) \leq F_n$ and $[\,]$ denote homology class "$z + B_n(F_*)$." $z = \beta_n(x)$ for some $x \in E_n$, by exactness at F_n. Let $\partial x = y \in E_{n-1}$. Now $\beta_{n-1}(y) = \beta_{n-1}(\partial x) = \partial(\beta_n x) = \partial z = 0$, since the square commutes. By the exactness at E_{n-1}, $y = \alpha_{n-1}(w)$ for some $w \in D_{n-1}$. Define $\partial_*[z] = [w] \in H_{n-1}D_*$. Need to check several things:

(1) Know w is a cycle so does indeed represent a class in A_* because $\alpha_{n-2}(\partial w) = \partial\alpha_{n-1}(w) = \partial y = \partial^2 x = 0$, and α_{n-2} is an injection so this implies $\partial w = 0$.

(2) Given z, what about choices of x, y, w? y is defined uniquely given x, and w given y (since α_{n-1} is an injection). If $\beta_n(x + x') = z$ is a different choice for x, then $\beta_n(x') = 0$ $\implies x' = \alpha_n(t)$ for some $t \in D_n$. Claim: $w + \partial t$ is new "w" (check uniqueness). This is okay since $[w + \partial t] = [w]$.

(3) Suppose $z + \partial v$ choosen as a different representative of $[z]$. Then $v = \beta_{n+1}(u)$ for some $u \in E_{n+1}$. So $x + \partial u$ is a choice of an element s.t. $\beta_n(x + \partial u) = z + \partial v$, and the new "$y$" is $\partial(x + \partial u) = \partial x$ ($\partial^2 = 0$) which is the same "y" as before.

(4) It is a homomorphism.

In order to show exactness of the long exact sequence, there are six cases of ker $\leq$im or ker $\geq$im to check.

(1) $\beta_* \circ \alpha_*$ (α_*, β_* homology maps) $= (\beta_* \circ \alpha_*)_*$ (α_*, β_* chain maps) $= 0_*$ by the exactness of $0 \to D_* \overset{\alpha_*}{\to} E_* \overset{\beta_*}{\to} F_* \to 0$. So $\ker \beta_* \geq \operatorname{im} \alpha_*$.

(2) To compute $\partial_* \circ \beta_*([s])$, start with $z = \beta_n(s)$ ($[z] = \beta_*[s] = [\beta_*(s)], s \in E_n$) then $y = \partial s = 0$ since s represents a cycle, and therefore $\partial_*[z] = 0$. So $\ker \partial_* \geq \operatorname{im} \beta_*$.

(3) To compute $\alpha_* \circ \partial_*([z])$, observe $\partial_*[z] = [w]$, and $\alpha_*\partial_*[z] = [y] = [\partial x] = 0$. So $\ker \alpha_* \geq \operatorname{im} \partial_*$.

(4) $\ker \beta_* \leq \operatorname{im} \alpha_*$: Assume we have a representative x of a cycle in $H_n(E)$ such that $\beta_*[x] = 0$. Ie. βx is a boundary in F, so $\beta x = \partial v$ for some $v \in F_{n+1}$. Since β_{n+1} surjective, there exists a $u \in E_n + 1$ such that $\beta(u) = v$. Observe that $x' = x - \partial u$ is still a cycle representing $[x]$, since it differs by a boundary, and that $\beta(x - \partial u) = \beta(x) - \beta\partial u = \beta x - \partial\beta u = 0 \in F_n$. By exactness of the s.e.s, there exists a $t \in D_n$ such that $\alpha(t) = x'$.

Claim t is a cylce in $H_n(D)$. Observe that $\alpha_{n-1}\partial t = \partial\alpha_n t = \partial x' = \partial x - \partial\partial u = 0$ (since x is a cycle and $\partial^2 = 0$). Since α_{n-1} is injective, $\partial t = 0$ and t is a cycle.

(5) $\ker \partial_* \leq \operatorname{im} \beta_*$: Assume we have a representative z of a cycle in $H_n(E)$ such that $\partial_*[z] = 0$. Ie. the image $w \in D_{n-1}$ of z is a boundary in $H_{n-1}(D)$, so there is a $t \in D_n$ so that $\partial t = w$. Want that there exists a cycle $x \in E_n$ such that $\beta(x) = z$. Recall that we have already constructed an $x \in E_n$ $\beta(x) = z$ as part of the construction ∂_*. For this x, $\alpha(w) = \partial(x)$, so let $x' = x - \alpha(t)$. Then $\partial x' = \partial x - \partial\alpha(t) = \partial x - \alpha\partial t = \partial x - \alpha w = 0$, so x' is a cycle. Observe $\beta(x') = \beta(x - \alpha(t)) = \beta(x) - \beta\alpha(t) = \beta(x) = z$, as desired.

(6) $\ker \alpha_* \leq \operatorname{im} \partial_*$: Assume we have a representative w of a cycle in $H_{n-1}(D)$ such that $\alpha_*[w] = 0$. Ie. $\alpha w = y = \partial x$ for some $x \in E_n$. Consider $z = \beta x$. In our construction of $\partial_* z$, we might as well choose x as the pre-image of z, so $\partial_*[z] = [w]$, assuming z is a cycle. But this is easily checked since $\partial z = \partial\beta x = \beta\partial x = \beta\alpha w = 0$. $\surd$

Mayer-Vietoris Sequence *If K is a simplicial complex where $K = A \cup B$ where A, B are subcomplexes, then there is a s.e.s.*

$$0 \to C_*(A \cap B) \stackrel{x \mapsto x \oplus -x}{\to} C_*(A) \oplus C_*(B) \stackrel{x \oplus y \mapsto x+y}{\to} C_*(A \cup B) \to 0$$

The two maps are actually $(i_{1*}, -i_{2*})$ and $j_{1*} + j_{2*}$ where $A \cap B \stackrel{i_1}{\hookrightarrow} A \stackrel{j_1}{\hookrightarrow} A \cup B = K$ and $A \cap B \stackrel{i_2}{\hookrightarrow} B \stackrel{j_2}{\hookrightarrow} A \cup B = K$ are simplicial maps. Consequently there exists a l.e.s. of homology

$$\to H_n(A \cap B) \to H_n(A) \oplus H_n(B) = H_*(C_*(A) \oplus C_*(B)) \to H_n(A \cup B) \stackrel{\partial_*}{\to} H_{n-1}(A \cap B) \to$$

Proof Obvious after a little thinking since everything is a free abelian group of simplices. $\surd$

Example Computing $H_*(\text{torus})$ directly but ugly from $0 \to C_2 \to C_1 \to C_0 \to 0$. $H_0 = C_0/\partial C_1$. $C_0 = \mathbf{Z}$-linear combinations of vertices but we care only modulo relations $\partial(\text{edge}) =$ end point-starting point. So $[5 \to 8] = [4 \to 9]$ since they differ by a boundary. Therefore $H_0 \cong \mathbf{Z}$ by concentrating charge at a single point. In general, $H_0 = \mathbf{Z}^n$, where n is the number of components of the 1-skeleton.

H_2 consists of linear combinations of 2-simplexes such that their boundary is 0. The boundary of n times a 2-simplex is n times each of the three boundary edges. Suppose we have a particular 2-simplex with weight n counter-clockwise in our linear combination. In order to cancel out the boundary, the three neighboring 2-simplexes must also have weight n counter-clockwise. This weight then must be extended to all the 2-simplexes. Check that this weighting is consistent on identified outer edges, so $H_2 \cong \mathbf{Z}$, generated by $\sum$ all 2-simplexes oriented counter-clockwise. (Observe for Klein bottle, it doesn't work so $H_2(K) = 0$. Still have to have weight n counter-clockwise on all 2-simplexes, but now boundary reinforces instead of cancelling).

To compute H_1, assume each edge has aquired a charge. May shift charge in a cycle off of one of the edges to the other two edges in that cycle. Eventually left with just dangling edges (which can't be part of a cycle) and two loops. Conclusion is $H_1 \cong \mathbf{Z} \oplus \mathbf{Z}$ (or at least contained in $\mathbf{Z} \oplus \mathbf{Z}$), generated by those two loops.

Example May also compute $H_*(\text{torus})$ by decomposing $T = A \cup B$ with A, B cylinders (prism shaped) with their intersection $A \cap B = C_1 \cup C_2$ two circles (triangles, actually, since made out of simplexes). A, B have homology $H_0(A) \cong H_0(B) \cong \mathbf{Z}$ generated by a vertex, $H_1(A) \cong H_1(B) \cong \mathbf{Z}$ generated by a cycle, and no other homology. To see this, either compute with brute force (as above) or show that projection to central circle is a "chain homotopy equivalence" (cf. cone TO DO-try this). For $A \cap B$, $H_0 \cong \mathbf{Z} \oplus \mathbf{Z}$ generated by p_1, p_2, a point on each circle, and $H_1 \cong \mathbf{Z} \oplus \mathbf{Z}$ generated by c_1, c_2, the two circles. Now apply Mayer-Vietoris:

$$\to H_{n+1}T \to H_n(A \cap B) \to H_n(A) \oplus H_n(B) \to H_n(T) \to H_{n-1}(A \cap B) \to$$

So $H_0 T \cong H_0 A \oplus H_0 B / \ker j_* = H_0 A \oplus H_0 B / \operatorname{im} i_* = \mathbf{Z} \oplus \mathbf{Z}/\mathbf{Z}(1,-1) \cong \mathbf{Z}$. This leaves us with

$$\begin{array}{l} 0 \to H_2 T \stackrel{\partial_*}{\hookrightarrow} H_1(A \cap B) \xrightarrow{i_{A*} \oplus -i_{B*}} H_1 A \oplus H_1 B \xrightarrow{j_{1*}+j_{2*}} \\ \to H_1 T \stackrel{\partial_*}{\to} H_0(A \cap B) \quad \xrightarrow{i_*} \quad H_0 A \oplus H_0 B \stackrel{j_*}{\twoheadrightarrow} H_0 T = \mathbf{Z} \to 0 \end{array}$$

So $H_2 T \cong \operatorname{im} \partial_*$ (by injectivity, since previous group is 0) $= \ker(i_{A*} \oplus -i_{B*}) = \langle c_1 - c_2 \rangle \cong \mathbf{Z}$. In the middle, $\operatorname{im} \partial_* = \ker i_* = \langle p_1 - p_2 \rangle \cong \mathbf{Z}$. Similarly

$$\begin{aligned} \ker \partial_* &= \operatorname{im}(j_{1*} + j_{2*}) = H_1 A \oplus H_1 B / \ker(j_{1*} + j_{2*}) \\ &= H_1 A \oplus H_1 B / \operatorname{im}(i_{1*} \oplus -i_{2*}) = \mathbf{Z} \oplus \mathbf{Z}/\mathbf{Z}(1,1) \cong \mathbf{Z} \end{aligned}$$

So we have $0 \to \mathbf{Z} \hookrightarrow H_1 T \twoheadrightarrow \mathbf{Z} \to 0$ (ie. a surjective map with isomorphims type of kernel known) $\Longrightarrow H_1 T \cong \mathbf{Z} \oplus \mathbf{Z}$. Why is this?

Lemma *If $0 \to A \to B \to C \to 0$ is exact and $C \cong \mathbf{Z}^n$ then $B \cong A \oplus C$. Not true in general that $B \cong A \oplus C$; consider $0 \to \mathbf{Z} \stackrel{\times 2}{\to} \mathbf{Z} \to \mathbf{Z}_2 \to 0$. Note that we know homology groups are all abelian.*

Definition A short exact sequence of abelian groups $0 \to A \stackrel{j}{\to} B \stackrel{p}{\to} C \to 0$ *splits* if there exists a map $i : C \to B$ such that $p \circ i = \mathrm{id}_C$.

Remark If the sequence splits, $B = j(A) \oplus i(C)$ (internal direct sum).

Proof (of lemma) If $C = \mathbf{Z}^n$, choose for each generator e_i an element $b_i \in B$ such that $p(b_i) = e_i$. Extend the map $e_i \mapsto b_i$ to a homomorphism $\mathbf{Z}^n \to B$. $\checkmark$

Singular Homology

Definition Let Δ^n be the standard n-simplex $\{(t_0, \ldots, t_n) \subseteq \mathbf{R}^N : \sum t_i = 1, 0 \leq t_i \leq 1\}$, with vertices $e_0, \ldots, e_n$. We have simplicial maps $s_i : \Delta^{n-1} \to \Delta^n$ taking $(e_0 \ldots e_{n-1}) \mapsto (e_0 \ldots \hat{e}_i \ldots e_n)$.

Definition For a fixed topological space X, a *singular n-simplex* is a continuous map $\Delta^n \to X$. Note that it can be very degenerate and it comes paramterised, eg. the two maps $\Delta^1 = I \to \mathbf{R}$ defined by $t \mapsto t$ and $t \mapsto t^2$ represent different singular 1-simplexes in $\mathbf{R}$.

Definition Define n^{th} *chain group* $C_n X$ to be the free abelian group generated by all such things (finite linear combinations). Define the *boundary map* $\partial : C_n \to C_{n-1}$ by $\partial(\sigma : \Delta^n \to X) = \sum_{i=0}^n (-1)^i (\sigma \circ s_i)$. Alternatively $\sum_{i=0}^n (-1)^i \sigma|(e_0 \ldots \hat{e}_i \ldots e_n)$, canonically identifying with Δ^{n-1} using s_i.

Lemma $\partial^2 = 0$

Proof Same as before. $\checkmark$

Remark Therefore we get cycles, bundaries, chain complexes, and homology defined by $H_n(X) = Z_n(X)/B_n(X)$.

Definition If $f : X \to Y$ is continuous, define $f_* : C_nX \to C_nY$ by $f_*(\sigma) = f \circ \sigma$ and extend linearly. Obviously have $(f \circ g)_* = f_* \circ g_*$, $\mathrm{id}_* = id_{C_*}$. Also f_* is a chain map $\partial f_* = f_*\partial$ (obvious). So we have induced maps $f_* : H_*X \to H_*Y$. Further if $X \xrightarrow{f} Y$ is a homeomorphims, then $f_* : H_*X \to H_*Y$ is an isomorphism. So homology groups are topological invariants.

Let us compute H_0 and H_1 for a topological space.

Theorem *If X is path-connected, $H_0 \cong \mathbf{Z}$ otherwise it is a free abelian on bases $\{$set of path components$\}$.*

Proof Singular 0-simplexes are points of X (=0-cycles). Singular 1-simplexes are paths. $\partial(\text{path } \gamma) = \gamma(1) - \gamma(0)$. Consequently,

$$H_0 = \frac{\text{Free Ab(pts of } X)}{\text{pts connected by a path}}$$

H_0(path-connected space) is non-trivial because there exists a surjective homomorphism called *augmentation*, $\epsilon : \frac{\text{Free Ab on pts of } X)}{\text{boundaries}} \to \mathbf{Z}$ by adding up the coefficients of points. $\surd$

Theorem *Let X be path-connected and fix a base point x_0. Then there exists a homomorphism $\theta : \pi_1(X, x_0) \to H_1(X)$ which is surjective, with kernel the commutator subgroup $\langle xyx^{-1}y^{-1} : x, y \in \pi_1\rangle$. In other words $H_1X \cong$ the abelianisation of $\pi_1(X, x_0)$.*

Proof Given α a loop based at x_0, define $\theta([\alpha]) = [\alpha]$ the homology class determined by α. This works since $\partial\alpha = \alpha(1) - \alpha(0) = 0$, so $[\alpha]$ is a 1-cycle.

(i) The constant loop maps to 0 in homology. Observe $(\Delta^1 \to x_0) = \partial(\Delta^2 \to x_0)$ $(= +(\Delta^1 \to x_0) - \Delta^1 \to x_0) + \Delta^1 \to x_0))$. So it is a boundary.

(ii) Given α, β homotopic rel $\{0, 1\}$, via homotopy $F : I^2 \to X$. Label the corners of I^2, the unit square, $PQRS$ and subdivide into two triangles, PQS and PRS. Crude picture:

$$\begin{array}{ccc} R \xrightarrow{\beta} S & & \\ {\scriptstyle x_0}\Big| \quad \nearrow \quad \Big|{\scriptstyle x_0} & \xrightarrow{F} & X \\ P \ \overrightarrow{\alpha}\ Q & & \end{array}$$

Let $F_1 : \Delta^2 \to PQS \xrightarrow{F} X$, and $F_2 : \Delta^2 \to PRS \xrightarrow{F} X$ be the maps from the standard 2-simplex. We will call these "$F|PQS, F|PRS$". Then (sign to make all orientations counter-clockwise?) $\partial(F_PQS - F_PRS) = F|QS - F|PS + F|PQ - F|RS + F|PS - F|PR$, where $F|RS$ means $\Delta^1 = (e_0, e_1) \to RS \xrightarrow{F} X$. Note that $F|QS$ and $F|PR$ are constant loops, $F|PQ$ is α, $F|RS$ is β, and the $F|PS$ terms cancel. Note that orientations and signs were chosen carefully so cause $F|PS$ to cancel, wouldn't have worked if we were left with $-F|PS - F|SP$. So $\partial(F_1 - F_2) = \partial(\Delta^2 \to x_0) + \alpha - \beta + \partial(\Delta^2 \to x_0)$ and $\alpha - \beta$ is a boundry $\Longrightarrow [\alpha] = [\beta]$ as homology classes.

(iii) To show θ is a homomorphism, suppose α, β are compatible paths. Claim: $\alpha + \beta - \alpha \cdot \beta$ is a boundary.

Proof Define a map from the standard 2-simplex $(e_0, e_1, e_2) \Longrightarrow$ the projection of the triangle onto the line (e_0, e_2) taking e_1 to the midpoint $\xrightarrow{\alpha\cdot\beta} X$. The composition is a map from the standard 2-simplex into X which traverses α through any line from e_0 to the line from e_1 to the midpoint of e_0e_2, and β through the line to e_2. The boundary of this singular 2-simplex is $\text{map}|e_1e_2 - \text{map}|e_0e_2 + \text{map}|e_0e_1 = \beta - \alpha\cdot\beta + \alpha$. If α, β are loops, then this says $\theta([\alpha]) + \theta([\beta]) - \theta([\alpha][\beta] = [\alpha\cdot\beta]) = 0$ in homology. $\surd$

Remark If α is a loop (so it represents a homology class), and α^{-1} is the homotopy inverse of α, then $[\alpha] + [\alpha^{-1}] = 0 \in H_1X$. Ie. $[\alpha^{-1}] = -[\alpha]$.

Proof $\alpha\cdot\alpha^{-1} \simeq e_{x_0}$ the constant loop at x_0 rel $\{0,1\}$. Result follows from (iii). $\surd$

(iv) To show θ is surjective, suppose we have a generic 1-cycle in X, $\sum_{i=1}^k \sigma_i : \Delta^1 \to X$ (a finite sum with possible duplication). Choose paths γ_p from x_0 to p for each $p \in \bigcup_i\{\sigma_i(0), \sigma_i(1)\}$. Define $\gamma_i = \gamma_{\sigma_i(0)}\cdot\sigma_i\cdot\gamma_{\sigma_i(1)}$ for each i. Consider the element $[\alpha] = [\gamma_1]\cdots[\gamma_k]$ in π_1. θ maps $[\alpha]$ to the homology class $[\sum(\gamma_{\sigma_i(0)}\cdot\sigma_i\cdot\gamma_{\sigma_i(1)})]$ (since α is a loop, $\theta([\alpha])$ is a cycle). Now for a given point p (an endpoint of some σ_i), group the γ's corresponding ones starting at p and those ending at p. Since our original singular 1-complex was a cycle, there will be an equal number in each category. Since $[\gamma_p][\bar{\gamma_p}] = 0$ in homology (regard it as $[\gamma_p\cdot\bar{\gamma_p}]$ which by the remark is 0). Cancelling all the γ_p's we have $\theta([\gamma_1][\gamma_2]\cdots[\gamma_k]) = [\sum\sigma_i]$ in homology.

(v) Want to show θ has kernel the commutator subgroup. It is clear the commutator subgroup is contined in the kernel since $xyx^{-1}y^{-1} \mapsto x + y - x - y = 0$. Need to show that if $[\alpha] \in \pi_1(X, x_0)$, $\theta([\alpha]) = 0$, then $[\alpha] = 0$ in π_1 mod the commutator subgroup. Suppose α is a loop based at x_0 which is a boundary in homology, ie. $\alpha = \partial\left(\sum_{i=1}^k \sigma_i\right)$ for some 2-simplexes σ_i. For each $p \in \bigcup_i\{\sigma_i(e_0)\sigma_i(e_1)\sigma_i(e_2)\}$ fix γ_p as before. For each edge, eg. $\sigma_i|(e_0,e_1)$ define $\gamma_{\sigma_i(e_0,e_1)} = \gamma_{\sigma_1(e_0)}\cdot\sigma_i|(e_0,e_1)\cdot\bar{\gamma_{\sigma_i(e_1)}}$. Let $\gamma_{\sigma_i} = \gamma_{\sigma_i(e_0,e_1)}\gamma_{\sigma_i(e_1,e_2)}\gamma_{\sigma_i(e_2,e_0)}$.

Now consider $\prod_{\sigma_i}\gamma_{\sigma_i}$ in $\pi_1/[\pi_1,\pi_1]$. First view each γ_{σ_i} as a product of three "γ_{σ_i} edges." Since, we are working in the abelianization of π_1, we may rearrange this product to cancel $\gamma_p, \bar{\gamma_p}$ pairs, leaving only $[\alpha] \in \pi_1/[\pi_1,\pi_1]$. However, instead we could homotop each $[\gamma_{\sigma_i}]$ to the constant loop individually. Therefore $[\alpha] = \prod_{\sigma_i}\gamma_{\sigma_i} = 0 \in \pi_1/[\pi_1,\pi_1]$. $\surd$

Theorem *If f, g are homotopic maps $X \to Y$, then $f_* = g_* : H_*X \to H_*Y$.*

Proof Given $F : X \times I \to Y$ a homotopy from f to g, wish to construct a chain homotopy $\{P_n : C_nX \to C_{n+1}Y\}$ satisfying $f_*(\sigma) - g_*(\sigma) = \partial P(\sigma) + P\partial(\sigma)\forall\sigma \in C_nX$. Given an element $[\sum_{i,n}\sigma_i^n] \in H_*X$, with $\sigma_i^n : \Delta^n \to X$, want to define $P(\sigma) \in C_{n+1}Y$. The idea is to divided the image of $F(\sigma(\Delta^n)\times I)$ into $n+1$-simplexes and let $P(\sigma) = \sum$ these.

For example, if $n = 0$, $\sigma^0(\Delta^0) = x$ a point in X. $F(x \times I)$ is a path in Y starting at $f(x)$ ending at $g(p)$. In this case we set $P(\sigma) = P(\Delta^0 \to x) = F \circ (\sigma \times \text{id}_I)$ thought of as a single singular simplex in C_1Y. If $n = 1$, so $\sigma^1(\Delta^1)$ is a path in X, subdivide $\sigma(\Delta^1)\times I$ into two singular simplexes $\sigma_0, \sigma_1 : \Delta^2 \to X \times I$. Then let $p(\sigma) = \sum_{i=1}^2 F\circ\sigma_i$, the sum of two singular simplexes.

Let $\Delta^n \times I$ be a prism sitting in $\mathbf{R}^N$ with $2n+2$ vertices $v_0 = e_0 \times \{0\}, \ldots, v_n = e_n \times \{0\}, w_0 = e_0 \times \{1\}, \ldots, w_n = e_n \times \{1\}$. Divide this prism with n hyperplanes through the points $\{v_0, \ldots, v_i, w_{i+1}, \ldots w_n\}$ into simplexes of the form $(v_0, \ldots, v_i, w_i, \ldots, w_n)$ (notice that this is an $n+1$-simplex, and the intersection of two adjacent simplexes of this form is the hyperplane indicated). Define operator $P : C_nX \to C_{n+1}Y$ by

$$P(\sigma) = \sum_{i=0}^n (-1)^i F \circ (\sigma \times \text{id}_I)|(v_0, \ldots, v_i, w_i, \ldots, w_n) \cong (e_0, \ldots, e_{n+1}) \text{ canonically}$$

and then extend linearly to C_nX. Compute:

$$\begin{aligned}\partial P(\sigma) &= \sum_{0\le j\le i\le n}(-1)^{i+j}F\circ(\sigma\times \mathrm{id}_I)|(v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n)+\\ &\quad \sum_{0\le i\le j\le n}(-1)^{i+j+1}F\circ(\sigma\times \mathrm{id})|(v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n)\end{aligned}$$

Recall $\partial\sigma=\sum_{j=0}^{n}(-1)^j\sigma|(e_0,\dots,\hat{e}_j,\dots,e_n$ so

$$\begin{aligned}P\partial\sigma &= \sum_{0\le i<j\le n}(-1)^{i+j}F\circ(\sigma\times \mathrm{id})|(v_0,\dots,v_i,w_i,\dots,\hat{w}_j,\dots,w_n)\\ &\quad +\sum_{0\le j<i\le n}(-1)^{i+j-1}F\circ(\sigma\times \mathrm{id})|(v_0,\dots,\hat{v}_j,\dots,v_i,w_i,\dots,w_n)\end{aligned}$$

Adding $P\partial\sigma$ and $\partial P\sigma$ and cancelling gives

$$\begin{aligned}P\partial\sigma+\partial P\sigma &= \sum_{i=0}^{n}F\circ(\sigma\times \mathrm{id})|(v_0,\dots,v_{i-1},w_i,\dots,w_n)-\\ &\quad \sum_{i=0}^{n}F\circ(\sigma\times \mathrm{id})|(v_0,\dots,v_i,w_{i+1},\dots,w_n)\\ &= F\circ(\sigma\times \mathrm{id})|(w_0,\dots,w_n)-F\circ(\sigma\times \mathrm{id})|(v_0,\dots,v_n)\\ &= g\circ\sigma-f\circ\sigma=g_*-f_*\end{aligned}$$

Therefore f_*, g_* are chain homotopic, and induce the same maps on homology. $\surd$

Cor *Homotopy equivalent spaces have the same homology.*

Proof Assume given $f: X\to Y$ and $g: Y\to X$ with $f\circ g\simeq \mathrm{id}_Y$ and $g\circ f\simeq \mathrm{id}_X$. Apply the homology functor, $f_*\circ g_*=\mathrm{id}_*$, an isomorphism, and $g_*\circ f_*=\mathrm{id}_*$ also an isomorphism. Therefore f_* and g_* are both isomorphisms. $\surd$

Mayer-Vietoris Sequence

We need to have a procedure for performing calculations. In simplicial theory, we have a s.e.s. $0\to C_*(A\cap B)\to C_*A\oplus C_*B\to C_*(A\cup B)\to 0$, and by the fundamental theorem of homological algebra gives a l.e.s. of homology. Now, using simplical homology, if we decompose a space X into two open sets A, B (instead of simplicial complexes), the above sequence is not exact. In simplicial homology, all we can say is that the sequence $0\to C_*(A\cap B)\xrightarrow{i_{A*}\oplus -i_{B*}} C_*A\oplus C_*B\xrightarrow{j_{A*}+j_{B*}} C_*(A\cup B)$ is exact. Observe that you can have a "too big" simplex which is contained in $A\cup B$ but not A or B individually. Compare this with the simplicial case where any simplex is contained in one of $A\setminus B, B\setminus A, A\cap B$.

So replace $C_*(A\cup B)$ by the image$(C_*A\oplus C_*B)=C_*(A+B)\le C_*(A\cup B)$ (remark: $C_*(A+B)$ is a chain complex since the boundary map behaves well on that subset, namely $\Im(\partial_{C_n(A\cup B)}|C_n(A+B)\le C_{n-1}(A+B))$. We have defined $C_*(A+B)$ such that $0\to C_*(A\cap B)\to C_*A\oplus C_*B\to C_*(A+B)\to 0$ is exact. Applying the fundamental theorem of homological algebra, we get a l.e.s. involving $H_*(A\cap B)$, $H_*A\oplus H_*B$, and "$H_*(A+B)$," the homology of the chain complex $C_*(A+B)$. Now the Mayer-Vietoris sequence for singular homology follows from the following lemma.

Lemma *There exists the inclusion* $i_* : C_*(A+B) \hookrightarrow C_*(A \cup B)$ *and a chain map* $p_* : C_*(A \cup B) \to C_*(A+B)$ *such that* $i_* \circ p_*$ *and* $p_* \circ i_*$ *are chain homotopic to identity. This implies* i_* *and* p_* *are isomorphisms between* $H_*(A+B)$ *and* $H_*(A \cup B)$.

Argue this by subdivision. Aim to replace "big" simplexes by linear combinations of smaller ones, using the Lebesgue Covering Lemma, to define p_*. Recall *barycentric subdivision* of a simplex $\sigma^n \subseteq \mathbf{R}^n$ is a complex whose simplexes are of the form $(\hat{\sigma_0}, \hat{\sigma_1}, \ldots, \hat{\sigma_k})$, $k \leq n$, where $\sigma_0 < \sigma_1 < \cdots < \sigma_k$ is a chain of proper inclusions of faces and if $\sigma_i = (a_0, \ldots, a_l)$, then $\hat{\sigma_i} = \frac{1}{l+1}\sum_{i=0}^{l} a_i$ is the barycenter of the face σ_i.

Fact: The maximum diameter of a simplex in the barycentric subdivision of σ is

$$\frac{n}{n+1}\operatorname{diam}\sigma$$

(easy to check). So repeatedly performing barycentric subdivision sends the diameter of the simplexes to 0.

We want to replace $\sigma : \Delta^n \to X$ with $\sum_i \sigma_i : \Delta^n \to X$ by subdividing in Δ^n. We start by doing this for *affine singular simplexes.* If $\sigma : \Delta^n \to \mathbf{R}^N$ is an affine map, and $a \in \mathbf{R}^N$, define *coning* $a \cdot \sigma : \Delta^{n+1} \to \mathbf{R}^N$ by $(e_0, \ldots, e_{n+1}) \overset{\text{affinely}}{\mapsto} (a, \sigma(e_0), \ldots, \sigma(e_n))$. Define inductively a subdivision map $\{S_n : C_n^{\text{aff}}\mathbf{R}^N) \to C_n^{\text{aff}}\mathbf{R}^N)\}$ by $S_0(\sigma : \Delta^0 \to \mathbf{R}^N)$ =itself (and extend linearly), $S_n(\sigma : \Delta^n \to \mathbf{R}^N) = \widehat{\sigma(\Delta^n)} \cdot S_{n-1}(\partial\sigma)$ (and extend linearly).

Remark: Coning has beeen extended to a linear map $C_n^{\text{aff}}\mathbf{R}^N) \to C_{n+1}^{\text{aff}}(\mathbf{R}^N)$. Eg. σ gets replaced by $\sigma_+ - \sigma_-$, where if σ maps e_0 to a and e_1 to b, σ_+ maps e_0 to $\widehat{ab}$ and e_1 to b, and σ_- maps e_0 to $\widehat{ab}$ and e_1 to a. Similarly, σ^n gets replaced by a sum of maps into the barycentric subdivision of $\sigma(\Delta^n)$ with appropriate signs. All we really need is to consider the effect on $\mathrm{id}_{\Delta^n} : \Delta^n \to \Delta^n \subseteq \mathbf{R}^N$.

Define for any topological space X, a map $S_n : C_nX \to C_nX$ by $S(\sigma : \Delta^n \to X) = \sigma_* S^{\text{aff}}(\Delta)$, ie. new σ as $\sigma \circ \mathrm{id}_{\Delta^n}$ and then subdividing by replacing the id_{Δ^n} with S^{aff}. Note $\sigma_* : C_n^{\text{aff}}(\Delta^n) \to C_n(X)$ is the usual chain map induced by σ.

Sub-Lemma *S is a chain map, ie.* $\partial S = S\partial$.

Proof

$$\begin{aligned}
\partial S_n(\sigma^n) &= \partial\sigma_* S_n^{\text{aff}}(\Delta) \\
&= \sigma_*\partial S_n^{\text{aff}}(\Delta) \\
&= \sigma_*\partial(\hat{\Delta^n} \cdot S_{n-1}^{\text{aff}}\partial\Delta) \\
&= \sigma_*(S_{n-1}^{\text{aff}}\partial\Delta - \hat{\Delta^n} \cdot \partial S_{n-1}^{\text{aff}}\partial\Delta) \text{ (used "}\partial(a\cdot\sigma) = \sigma - a\partial\sigma\text{")}
\end{aligned}$$

Now proceed by induction: assume $\partial S_{n-1} = S_{n-1}\partial$ ($\partial S_0 = S_0\partial$ is clear), so $\partial S_{n-1}\partial = \partial^2 S_{n-1} = 0$, and

$$\begin{aligned}
\partial S_n(\sigma^n) &= \sigma_* S_{n-1}^{\text{aff}}\partial\Delta \\
&= S_{n-1}(\sigma_*\partial\Delta) \text{ (note not affine)} \\
&= S_{n-1}\partial\sigma_*\Delta = S_{n-1}\partial\sigma
\end{aligned}$$

which completes the induction and the proof. $\surd$

Now we need to show that S is chain homotopic to $\mathrm{id}_{C_n(X)}$. Define $T_n^{\text{aff}} : C_n^{\text{aff}}\mathbf{R}^N) \to C_{n+1}^{\text{aff}}(\mathbf{R}^N)$ by $T_n^{\text{aff}}(\sigma) = \sigma(\hat{\Delta^n}) \cdot (\sigma - T_{n-1}^{\text{aff}}(\partial\sigma))$ and letting $T_0^{\text{aff}}(\sigma)$ be the constant map

$\Delta^1 \to \sigma(\Delta^0)$. Define for any X, $T : C_n X \to C_{n+1} X$ by $T(\sigma) = \sigma_* T^{\text{aff}}(\Delta \xrightarrow{\text{id}} \Delta)$. Eg. $T(\text{pt} \to p \in X) = (\Delta^1 \to p \in X)$, $T(\Delta^1 \xrightarrow{\gamma} \text{path} \subseteq X) = (\Delta^2 \to \text{path})$ such that $e_0 \mapsto \gamma(e_0)$, $e_1 \mapsto \gamma(\widehat{e_0 e_1})$, $e_2 \mapsto \gamma(e_1)$. Want to show $\partial T + T\partial = id - S$. The picture is really of a prism with one end subdivided mapping into X by projecting to the base of the prism and then using σ.

Sub-Lemma *(T is not a chain map!) T is a chain homotopy with $\partial T + T\partial = \text{id} - S$.*

Proof

$$\begin{aligned}
\partial T(\sigma) &= \partial \sigma_*(\hat{\Delta^n} \cdot (\Delta - T^{\text{aff}}_{n-1}\partial\Delta)) \\
&= \sigma_* \left[(\Delta - T^{\text{aff}}_{n-1}\partial\Delta) - \hat{\Delta^n} \cdot \partial(\Delta - T^{\text{aff}}_{n-1}\partial\Delta)\right] \\
&= \sigma_* \left[(\Delta - T^{\text{aff}}_{n-1}\partial\Delta) - \hat{\Delta^n} \cdot ((\text{id}_{C^{\text{aff}}_{n-1}(\mathbf{R}^N)} - \partial T^{\text{aff}}_{n-1})(\partial\Delta))\right]
\end{aligned}$$

Now applying induction, we may assume $\text{id} - \partial T^{\text{aff}}_{n-1} = S^{\text{aff}} + T^{\text{aff}}\partial$, so

$$\begin{aligned}
\partial T(\sigma) &= \sigma - T\sigma\partial\Delta - \sigma_*(\hat{\Delta^n} \cdot (S^{\text{aff}} + T^{\text{aff}}\partial)(\partial\Delta)) \\
&= \sigma - T\partial\sigma - \sigma_*(\hat{\Delta^n} \cdot S^{\text{aff}}_{n-1}\partial\Delta) \\
&= \sigma - T\partial\sigma - \sigma_* S^{\text{aff}}_n \Delta \\
&= \sigma - T\partial\sigma - S\sigma
\end{aligned}$$

Therefore $\partial T + T\partial = \text{id} - S$. $\surd$

We are now equiped to prove the Lemma, since we are in a position to construct p such that $C_*(A+B) \underset{p}{\overset{i}{\rightleftarrows}} C_*(A \cup B)$ is a chain homology equivalence.

Proof Given a singular simplex $\sigma : \Delta^n \to A \cup B$, use the Lebesgue covering for $\Delta^n = \sigma^{-1}(A) \cup \sigma^{-1}(B)$, to find the least $m > 0$ such that all of the simplexes in $S^m(\sigma)$ are indiviually in A or in B. Such an m exists because on repeatedly barycentrically subdividing, the diamter of simplexes goes to 0, so we could set m to be the least such that $\left(\frac{n}{n+1})\right) < \delta$. Define $D(\sigma) = \sum_{i=0}^m TS^i(\sigma)$. Compute

$$\begin{aligned}
\partial D(\sigma) &= \sum_{i=0}^{m-1} \partial T S^i \sigma \\
&= \sum_{i=0}^{m-1} (\text{id} - S - T\partial)(S^i\sigma) \\
&= \sum_{i=0}^{m-1} S^i\sigma - \sum_{i=1}^{m} S^i\sigma - \sum_{i=0}^{m-1} TS^i\partial\sigma \text{ (S is a chain map)} \\
&= \sigma - S^m\sigma - \text{“}D(\partial\sigma)\text{”}
\end{aligned}$$

However $\sum_{i=1}^m TS^i(\partial\sigma)$ is not actually $D(\partial\sigma)$: it probably has too many terms, since the "m" needed for the simplexes in ∂S will in general be less than the "m" for σ. So let $i \circ p(\sigma) = S^m\sigma - (\sum_{i=1}^m TS^i(\partial\sigma) - D(\partial\sigma))$. Actually, this defines $p : C_*(A \cup B) \to C_*(A+B)$ because i is an injection. Finally observe

1. p is a chain map since $i(\partial p)(\sigma) = \partial(i \circ p)(\sigma) = \partial\sigma - \partial D\partial\sigma$ and $i(p\partial)(\sigma) = \partial\sigma - \partial D\partial\sigma - D\partial\partial\sigma = \partial\sigma - \partial D\partial\sigma = i(\partial p)(\sigma)$. i injective $\Longrightarrow \partial p = p\partial$.

2. $p \circ i = \mathrm{id}_{C_*(A+B)}$ because $i(\sigma)$ is in $C_*(A+B)$ so it does not need to be subdivided, ie. $p = S^0\sigma = \sigma$.

3. $i \circ p - id_{C_*(A\cup B)} = \partial D + D\partial$ since $\partial D(\sigma) = \sigma - i \circ p(\sigma) - D\partial(\sigma)$. $\surd$

Mayer-Vietoris Theorem *If $X = A \cup B$ (open sets), we have a long exact sequence of singular homology groups*

$$\cdots \to H_{n+1}(X) \to H_n(A \cap B) \to H_nA \oplus H_nB \to H_nX(\cong H_n(A+B)) \to H_{n-1}(A \cap B) \to \cdots$$

Reduced Homology

Occasionally, the fact that $H_*(\mathrm{pt}, \mathrm{ball}, \mathrm{contractible}) = \begin{cases} \mathbf{Z} & \dim 0 \\ 0 & \text{otherwise} \end{cases}$ is annoying. Consequently introduce *augmentation* $\epsilon : C_0X \to \mathbf{Z}$ defined by $\epsilon(\sum n_i\sigma_i) = \sum n_i$. This gives an *augmented* chain complex $\to C_2X \xrightarrow{\partial} C_1X \xrightarrow{\partial} C_0X \xrightarrow{\epsilon} \mathbf{Z} \to 0$. Observe that if $\sigma \in C_1X$, then $\epsilon\partial(\sigma) = \epsilon(+\sigma(1) - \sigma(0)) = 0$.

Definition Define *reduced homology* $\tilde{H}_*(X)$ as the homology of the augmented chain complex. Observe $\tilde{H}_i(X) \cong H_i(X)$ if $i > 0$. For $i = 0$, $H_0(X) = C_0X/\mathrm{im}\partial_1 \overset{\epsilon}{\twoheadrightarrow} \mathbf{Z}$. The kernel of this map $H_0 \to \mathbf{Z}$ is just $\ker\epsilon/\mathrm{im}\,\partial_1 = \tilde{H}_0(X)$. In other words $0 \to \tilde{H}_0 \to H_0 \to \mathbf{Z} \to 0$. Therefore $H_0X \cong \tilde{H}_0X \oplus \mathbf{Z}$. Also, observe $\tilde{H}_0(\mathrm{pt}) = 0$ so $\tilde{H}_*(\mathrm{contractible}) = 0$ ($H_i(\mathrm{pt})$ is still zero for $i \leq -1$).

Theorem $H_*(S^{n\geq 1}) = \begin{cases} \mathbf{Z} & \mathit{dim}\ 0, n \\ 0 & \mathit{otherwise} \end{cases}$ $\left[H_*S_0 = \begin{cases} \mathbf{Z} \oplus \mathbf{Z} & \mathit{dim}\ n \\ 0 & \mathit{otherwise} \end{cases} \right]$

Proof Induction on n. Start from $H_1S^1 \cong \mathbf{Z}$ (the abelianisation of π_1). Split $S^n = A \cup B$, for $A = S^n - \{N\}$, $B = S^n - \{S\}$. Note that $A \cap B \simeq S^{n-1}$ by a deformation retract to the equator. Use Mayer-Vietoris sequence with reduced homology (where it still works) and $A \simeq B \simeq \mathrm{pt}$ so

$$\cdots \to \tilde{H}_nA \oplus \tilde{H}_nB = 0 \to \tilde{H}_nS^n \to \tilde{H}_{n-1}S^{n-1} \to \tilde{H}_{n-1}A \oplus \tilde{H}_{n-1}B = 0 \to \cdots$$

Conclude $\tilde{H}_nS^n \simeq \tilde{H}_{n-1}S^{n-1}$, in fact $\tilde{H}_iS^n \simeq \tilde{H}_{i-1}S^{n-1}$. Since $\tilde{H}_*S^1 = \begin{cases} \mathbf{Z} & \dim 1 \\ 0 & \text{otherwise} \end{cases}$ we get the desired result after converting the result out of reduced homology. $\surd$

Corollary *There does not exist a retraction $B^n \to S^{n-1}$.*

Proof If there was a retraction 'r' such that $r \circ \mathrm{inclusion} = \mathrm{id}_{S^{n-1}}$ then we would have

$$\begin{array}{ccccc} \tilde{H}_{n-1}S^{n-1} & \xrightarrow{i_*} & \tilde{H}_{n-1}B^n & \xrightarrow{r_*} & \tilde{H}_{n-1}S^{n-1} \\ \mathbf{Z} & \to & 0 & \to & \mathbf{Z} \end{array}$$

So $i_* \circ r_*$ can not be an isomorphism. $\surd$

Brower Fixed Point Theorem *Any continuous map $B_n \to B_n$ has a fixed point.*

Proof If $f(x) \neq x \forall x \in B_n$, with f a continuous map $B_n \to B_n$, then define $r : B^n \to S^{n-1}$ by $r(x) = (f(x) - x)/|f(x) - x|$. $r(x)$ is continuous since $|f(x) - x| > 0$, contradicting the corollary above. $\surd$

Picture An explicit picture of a generator for H_nS^n: Realise S^n as an "octahedron," *cross-polytope* in $\mathbf{R}^{n+1}$, namely the boundary of the convex hull of the vectors

$\{\pm e_0, \pm e_1, \ldots, \pm e_n\}$. This is $\cong S^n$ by radial projection. Identify each simplicial face $(\epsilon_0 e_0, \ldots, \epsilon_n e_n)$ with each $e_i = \pm 1$ (of the 2^{n+1} possibilities) with Δ^n the standard simplex using this ordering of vertices. Claim $Z_n \equiv \sum_{e_0,\ldots,e_n} (\prod_{i=0}^n \epsilon_i)\,(\Delta^n \to (\epsilon_0 e_0, \ldots, \epsilon_n e_n))$ is a cycle representing a generator of $H_n S^n \cong \mathbf{Z}$.

Proof (by induction) Check case $n = 1$ with a picture or use reduced homology and observe $Z_0 = (\Delta^0 \to +e0) - (\Delta^0 \to -e0)$ generates $\tilde{H}_0 S^0 \cong \mathbf{Z}$. Will use induction to show the boundary map $\to H_n S^n \overset{\partial'_*}{\to} H_{n-1}(A \cap B \simeq S^{n-1}) \to \cdots$ carries $[Z_n]$ to $[Z_{n-1}]$. Split $Z_n = Z_n^+ Z_n^-$ where Z_n^+ has all simplices with $\epsilon_n = +1$ and Z_n^- has the rest.

$$\partial Z_n^+ = \partial(\sum_{e_0,\ldots,e_{n-1}} (\prod_{i=0}^{n-1} \epsilon_i)(\pm 1)\,(\Delta \to (\epsilon_0 e_0, \ldots, \epsilon_{n-1} e_{n-1}, e_n)) = (-1)^n Z_{n-1}$$

where everything cancels except the term which comes from "hatting" e_n. Also $\partial Z_n^- = -(-1)^n Z_{n-1}$ so $\partial Z_n = 0$ and Z_n is a cycle. Now compute $\partial_*[Z_n \in H_n S^n] \in H_{n-1} S^{n-1}$, where $\partial_* : H_n S^n \overset{\simeq}{\to} H_{n-1} S^{n-1}$:

$$\begin{array}{ccccccc}
 & & & Z_n^+ \oplus Z_n^- \in & & Z_n \in & \\
0 \to & C_n S^{n-1} & \longrightarrow & C_n A \oplus C_n B & \overset{j_{A*}+j_{B*}}{\longrightarrow} & C_n S^n = C_n(A+B) & \to 0 \\
 & \downarrow & & \downarrow \scriptstyle{\partial \oplus \partial} & & \downarrow & \\
 & C_{n-1} S^{n-1} & \overset{i_{A*} \oplus - i_{B*}}{\longrightarrow} & C_{n-1} A \oplus C_{n-1} B & \longrightarrow & C_{n-1} S^n & \to 0 \\
 & (-1)^n Z_{n-1} & & (-1)^n Z_{n-1} \oplus -(-1)^n Z_{n-1} & & &
\end{array}$$

So $\partial_*[Z_n] = (-1)^n [Z_{n-1}]$ by induction down to $n = 1$ they are all generators. Remark: could use $\sum \pm(\Delta^n \to$ faces of $\Delta^{n+1})$ but that would be more tricky. $\surd$

Remarks Mayer-Vietoris sequence can be used to show that for a finite CW-complex: H_* is finitely-generated in each dimension, $H_n = 0$ for $n >$dimension of complex, and Mayer-Vietoris sequence is the analogue of Van Kampen's theorem (consider computing $H_*(\text{space} \cup \text{cell})$ by Mayer-Vietoris).

Definition If $f : S^n \to S^n$ is a continuous map, $f_* : H_n S^n = \mathbf{Z} \to H_n S^n = \mathbf{Z}$, its *degree* is $f_*(1) \in \mathbf{Z}$ (since f_* is a homomorphism $\mathbf{Z} \to \mathbf{Z}$). To identify the degree of f, choose a generator and see where it goes under f. For all $[g] \in H_n S^n$ define $f_*[g] = \deg f \cdot [g]$.

Remark Homotopic maps have the same degree.

Example 1) The identity id_{S^n} has degree $+1$ (naturality).

Example 2) The constant map $S^n \to \mathrm{pt} \in S^n$ has degree 0. It factors $H_n S^n \to H_n \mathrm{pt} \to H_n S^n$; $\mathbf{Z} \to 0 \to \mathbf{Z}$. Therefore the identity and the constant map are not homotopic.

Example 3) Any homeomorphism f has degree ± 1 (homeomorphisms induce isomorphisms on homology) "f preserves or reverses orientation."

Lemma *The degree of the antipodal map of S^n is $(-1)^{n+1}$.*

Proof Explicitly it sends Z_n to $(-1)^{n+1} Z_n$, recall

$$Z_n = \sum_{\epsilon_0,\ldots,\epsilon_n} \left(\prod \epsilon_i\right) (\Delta \to (\epsilon_0 e_0, \ldots, \epsilon_n e_n))$$

Alternatively, express the antipodal map as the composite of $(n+1)$ reflections. Can check each reflection has degree -1 explicitly. $\surd$

Remark $\deg(f \circ g) = \deg f \times \deg g$ by functoriality.

Hairy Ball Theorem *There does not exist a nowhere-zero continuous tangent vector field on an even-dimensional* S^n

Proof Suppose there exists such a field $v(x)$, then we can find a unit vector field $v(x)/\|v(x)\|$. Define $H(x,t) = (\cos\pi t)x + (\sin\pi t)v(x)/\|v(x)\|$. This is a continuous homotopy between id_{s^n} and the antipodal map. Contradiction since $\deg\mathrm{id} = +1$ and $\deg\mathrm{antip} = -1$. $\surd$

Remark On odd-dimensional spheres there exist such fields by regarding $S^{2n-1} \subseteq \mathbf{C}^n$. Then id is homotopic to the antipodal map by a complex rotation: $H(z,t) = e^{i\pi t}z$. The derivative of this homotopy, namely $z \mapsto i\pi Z$, is a nonzero tangent vector field.

Theorem *Any continuous map* $S^n \to S^n$ *with* n *even and with degree not* -1 *has a fixed point.*

Proof Suppose not, then define a homotopy from f to antipodal map:

$$H(x,t) = \frac{(1-t)f(x) + t(-x)}{\|(1-t)f(x) + t(-x)\|}$$

The denominator is never zero since $f(x) \neq x \forall x$. This is a contradiction since $\deg f \neq \deg\mathrm{antip}$. $\surd$

Theorem *If a group acts continuously and freely (ie. without fixed points) on* S^{even} *then* $G = 1$ *or* $G = \mathbf{Z}_2$*.*

Proof No group-element except id has a fixed point, so the homeomorphisms induced by these group elements all have degree -1. So the homomorphic map $G \stackrel{\deg}{\to} \{\pm 1\}$ (with multiplication as the group operation) has trivial kernel so $G \leq \mathbf{Z}_2$. $\surd$

Relative Homology

Let X be a space and A be a subspace. Define $C_n(X,A)$ (relative n-chains) $= C_n(X)/C_n(A)$ (note $C_n(A)$ is included inside $C_n(A)$). Observe that $\partial : C_nX \to C_{n-1}X$ restricts to $C_nA \to C_{n-1}A$ so we get an induced ∂ on $C_*(X,A)$ making it a chain complex. Consequently we have $Z_n(X,A)$, $B_n(X,A)$, and $H_n(X,A)$ the *Relative Homology*. An element of $Z_n(X,A)$ [ie. a representitive of a relative homology class] is represented by an n-chain in C_nX whose boundary consists of simplexes in A. Note that there are simplicial complexes which are not a cycle in X, but whose boundary lies entirely in C_*A. $B_n(X,A)$ consists of cosets $B_n(X) + C_n(A)$ in $C_n(X,A) = C_nX/C_nA$. Automatically we get the s.e.s. $0 \to C_*A \to C_*X \to C_*(X,A) \to 0$. Hence we get a l.e.s. of homology

$$\to H_n(A) \to H_n(X) \to H_n(X,A) \stackrel{\partial_*}{\to} H_{n-1}(A) \to$$

In fact $\partial_*[z] = [\partial z]$. Note we know $z \in C_n(X)$ with $\partial z \in A$. Observe ∂z is a cycle since $\partial^2 = 0$. Since $z \in C_n(X)$ rather than $z \in C_n(A)$, ∂z is not automatically a boundary in $C_n(A)$.

Words of wisdom "Everybody has a copy of Spanier which they keep on their highest shelf in case of emergencies."

Definition A *map of pairs* is a continuous function $f : (X,A) \to (Y,B)$, meaning $f : X \to Y$ but with $f(A) \subseteq f(B)$.

Simple facts Maps of pairs induce morphisms of relative homology. It is all functorial. Homotopic maps induce the same morphisms by homotopies $(X \times I, A \times I) \to (Y, B)$. Reduced relative homology is defined so that the augmented chain complexes still satisfy the s.e.s:

$$\begin{array}{ccccccccc}
0 \to & C_1 A & \to & C_1 X & \to & C_1(X,A) & \to 0 \\
& \downarrow \partial & & \downarrow \partial & & \downarrow \partial & \\
0 \to & C_0 A & \to & C_0 X & \to & C_0(X,A) & \to 0 \\
& \downarrow \epsilon & & \downarrow \epsilon & & \downarrow \epsilon & \\
0 \to & \mathbf{Z} & = & \mathbf{Z} & \to & 0 & \\
& \downarrow & & \downarrow & & \downarrow & \\
& 0 & & 0 & & 0 &
\end{array}$$

Ie. we augment by zero for reduced homology and thus $\tilde{H}_n(X, A) = H_n(X, A) \forall n$. Consequently the l.e.s. still works for reduced homology

$$\to \tilde{H}_n(A) \to \tilde{H}_n(X) \to \tilde{H}_n(X, A) \to \tilde{H}_{n-1}(A) \to$$

and $0 \to \tilde{C}(A) \to \tilde{C}(X) \to \tilde{C}(X, A) \to 0$ is still exact $\forall n$.

Excision *(1) If $U \subseteq A \subseteq X$ and $\bar{U} \subseteq \operatorname{Int} A$, then the inclusion $(X-U, A-U) \hookrightarrow (X, A)$ induces an isomorphism $H_*(X-U, A-U) \to H_*(X, A)$. (2) If $X = \operatorname{Int} A \cup \operatorname{Int} B$, then $(B, A \cap B) \hookrightarrow (X, A)$ induces an isomorphism $H_*(B, A \cap B) \to H_*(X, A)$.*

Proof (1) $\iff$ (2) By setting $U = X - B$ so $X - U = B$, $A - U = A \cap B$, the condition $\bar{U} \subseteq \operatorname{Int} A$ is equivalent to $X \bar{-} B \subseteq \operatorname{Int} A \subseteq \operatorname{Int} B \cup \operatorname{Int} A = X$. $\surd$

Proof of (2). Recall we had $C_n(A+B) \leq C_n(X)$ and a diagram of short exact sequences

$$\begin{array}{ccccccc}
0 \to & C_n A & \to & C_n(A+B) & \to & C_n(A+B)/C_n A & \to 0 \\
& \| & & \downarrow \text{natural inclusion} & & \downarrow & \\
0 \to & C_n A & \to & C_n X & \to & C_n(X)/C_n A & \to 0
\end{array}$$

This gives us two l.e.s. with commuting maps between them:

$$\begin{array}{ccccccccc}
H_n A & \to & H_n(A+B) & \to & H_n\left(\frac{C_*(A+B)}{C_* A}\right) & \to & H_{n-1} A & \to & H_{n-1}(A+B) \\
\downarrow = & & \downarrow \simeq & & \downarrow & & \downarrow = & & \downarrow \simeq \\
H_n A & \to & H_n X & \to & H_n(X, A) & \to & H_{n-1} A & \to & H_{n-1} X
\end{array}$$

5-Lemma *If you have a diagram like this of modules, with the top and bottom rows exact, and the outer four vertical maps are isomorphisms, then the middle vertical map is an isomorphism as well (proof by diagram chasing).*

We conclude that $H_n\left(\frac{C_*(A+B)}{C_* A}\right) \cong H_*(X, A)$. Observe however that $C_*(A+B)/(C_* A)$ is the free abelian group generated by all singular simplex in B not in A, which is also describes $C_n(B)/C_n(A \cap B)$. Consequently its homology is isomorphic to $H_*(B, A \cap B)$. $\surd$

Cellular Homology

Let X be a CW complex.

Lemma *(i)* $H_i(X^n, X^{n-1}) = \begin{cases} 0 & \text{if } i \neq n \\ \mathbf{Z}^{\#\ n\text{-cells}} & \text{if } i = m \end{cases}$ *(note that B^n with its boundary identified is S^n). (ii) $H_i(X^n) = 0$ if $i > n$. (iii) $X^n \subseteq X$ induces an isomorphism $H_i(X^n) \to H_i(X)$ for $i < n$.*

Proof (i) $X^n/X^{n+1} = \bigvee S^n$ ($\bigvee$ means glue together at a point), one corresponding to each n-cell (WARNING: logic is suspect since it does not address $H(A/B) \neq H(A,B)$). Using Mayer-Vietoris $\tilde{H}_i(\bigvee S^n) \cong \bigoplus \tilde{H}_i(S^n)$. In fact there is an explicit isomorphism since the characteristic maps $\Phi_i : (B_i^n, \partial B_i^n) \to (X^n, X^{n-1})$ induce an isomorphism $(\coprod \Phi_i)_* : \bigoplus \tilde{H}_n(B_i^n, \partial B_i^n) \to tildeH_n(X^n, X^{n-1})$.

(ii) The l.e.s. of relative homology gives (since $i > n$, $i \geq 1$)

$$H_{i+1}(X^n, X^{n-1}) = 0 \to H_i(X^{n-1}) \to H_i(X^n) \to H_i(X^n, X^{n-1}) = 0$$

Zeros follow from part (i) and $i > n$. Since $i \geq 1$, $H_i(X^n) \cong H_i(X^{n-1}) \cong \cdots \cong H_i(X^0) = 0$.

(iii) Use the same sequence, except reindex to get

$$H_{i+1}(X^{n+1}, X^n) = 0 \to H_i(X^n) \to H_i(X^{n+1}) \to H_i(X^{n+1}, X^n) = 0$$

Zeros are from part (i) and $i < n$. Therefore $H_i(X^n) \cong H_i(X^{n+1}) \cong H_i(X^{n+2}) \cong \cdots$. Need to show this equals $H_i(X)$. Consider $[z]$ some element of $H_i(X)$; z is a finite linear combination of singular simplexes, so "im (z)" is compact. Since a compact set meets only finitely many cells, $\operatorname{im}(z) \subseteq X^N$ for some sufficiently large N, then z represents an element in $H_i(X^N)$ which maps to $[z]$ in $H_i(X)$. Similarly, if $[z_1], [z_2] \in H_i(X^n)$ map to the same class in $H_i(X)$, the boundary they differ by can be represented inside X^N for some N, ie. $[z_1] = [z_2]$ in $H_i(X^N) \Longrightarrow$ they are equal in $H_i(X^n)$. $\surd$

Define Cellular Complex by considering l.e.s. for (X^{n+1}, X^n), (X^n, X^{n-1}), and (X^{n-1}, X^{n-2}):

$$\begin{tikzcd}
 & & & & 0 = H_n(X^{n+1},X^n) & & \\
{} \arrow[r, "\text{exact}"] & 0 = H_n(X^{n-1}) \arrow[dr] & & H_nX \cong H_n(X^{n+1}) \arrow[ur] & & & \\
 & & H_n(X^n) \arrow[ur, "i_n"] \arrow[dr, "j_n"] & & & & {} \\
 & H_{n+1}(X^{n+1},X^n) \arrow[ur, "\partial_{n+1}"] \arrow[rr, "d_{n+1}=j_n\circ\partial_{n+1}"'] & & H_n(X^n,X^{n-1}) \arrow[rr, "d_n=j_{n-1}\circ\partial_n"] \arrow[dr, "\partial_n"'] & & H_{n-1}(X^{n-1},X^{n-2}) \arrow[ur, "exact"] & \\
{} \arrow[ur, "\text{exact}"] & & & & H_{n-1}(X^{n-1}) \arrow[ur, "j_{n-1}"'] & & \\
 & & & H_{n-1}(X^{n-2}) = 0 \arrow[ur] & & &
\end{tikzcd}$$

Definition $C_n^{\text{cell}}(X) = H_n(X^n, x^{n-1})$, the free abelian group on n-cells, identified with $\mathbf{Z}^{\#\ \text{cells}}$ via characteristic maps $(\Phi_i^n)_*$ and a choice of generator for each $H_n(B_i^n, \partial B_i^n)$. Define $d_n : C_n^{\text{cell}} \to C_{n-1}^{\text{cell}}$ by $d_n = j_{n-1} \circ \partial_n$ as in the picture.

Claim *C_*^{cell} is a chain complex.*

Proof $d_n \circ d_{n+1} = j_{n-1} \circ \partial_n \circ j_n \circ \partial_{n+1}$. $\partial_n \circ j_n = 0$ since they are two adjacent maps in an exact sequence. $\surd$

Theorem $H_*(C_*^{\text{cell}}(X)) \cong H_*^{\text{sing}}(X)$.

Proof

$$\begin{array}{rcl} H_n^{\text{sing}}(X) & \overset{\text{lemma}}{\cong} & H_n^{\text{sing}}(X^{n+1}) \cong H_n(X^n)/\ker i_n \\ & \overset{\text{exactness}}{=} & H_n(X^n)/\text{im}\,\partial_{n+1} \\ & \overset{j_n \text{ injects}}{=} & \text{im}\, j_n/\text{im}\,(j_n \circ \partial_{n+1}) \\ & \overset{\text{exactness/rewriting}}{=} & \ker \partial_n/\text{im}\, d_{n+1} \\ & \overset{j_{n-1} \text{ injects}}{=} & \ker j_{n-1} \circ \partial_n/\text{im}\, d_{n+1} \\ & = & \ker d_n/\text{im}\, d_{n+1} = H_n^{\text{cell}}(X) \end{array}$$

$\surd$

Remark If K is a simplicial complex, then $H^{\text{simplicial}}(K) \cong H^{\text{cell}}(|K|) \cong H^{\text{sing}}(|K|)$. The first equality is shown by simply viewing K as a CW-complex, all that remains is to prove that d_n of the cellular complex equals ∂_n of the simplicial chain complex. The second equality is from the previous theorem.

Visualization A basis (1-simplex) element of $H_n(X^n, X^{n-1}) \cong C_n^{\text{sing}}(K)$ is represented by a singular simplex $[\Delta^n \to i^{\text{th}}$ n-simplex in $X^n]$ (ie. $(\coprod \Phi_i)_* : \bigoplus H_n(\Delta^n, \partial\Delta^n) \to H_n(X^n, X^{n-1})$ is an isomorphism). Then $j_{n-1} \circ \partial_n([\Delta^n \to \text{simplex in } X]) = \partial^{\text{sing}}(\text{this map})]$ a class inside X^{n-1}.

Example To compute $H_*(S^1 \times S^2)$ consider the CW-complex decomposition formed by choosing $p \in S^1$ and $q \in S^2$, specifically $S^1 \times S^2 = (\{p\} \times \{q\}) \cup ((S^1 - \{p\}) \times \{q\}) \cup (\{p\} \times (S^2 - \{q\})) \cup ((S^1 - \{p\}) \times (S^2 - \{q\}))$ (a 0-cell, a 1-cell, a 2-cell, and a 3-cell respectively). Compute the chain complex $0 \to \mathbf{Z} \overset{d_3}{\to} \mathbf{Z} \overset{d_2}{\to} \mathbf{Z} \overset{d_1}{\to} \mathbf{Z} \to 0$. Claim d_i are zero, and therefore $H_* = \mathbf{Z}$ in each dim.

Example For the torus, consider the decomposition into a point p, edges e_1, e_2 attached to that point, and face (2-cell) attached via $e_1 e_2 e_1^{-1} e_2^{-1}$. Chain groups are really $H_0(p)$, $H_1(I_1, \partial I_1) \oplus H_1(I_2, \partial I_2)$, and $H_2(I^2, \partial I^2)$. The (cellular) boundary of $H_1(I_j, \partial I_j)$ is $+p - p = 0$. For $H_2(I^2, \partial I^2)$ the boundary is $+e_1 + e_2 - e_1 - e_2 = 0$. We get the chain map

$$0 \to \langle f \rangle \cong \mathbf{Z} \to \langle e_1, e_2 \rangle \cong \mathbf{Z} \oplus \mathbf{Z} \to \langle p \rangle \cong \mathbf{Z} \to 0$$

with $\partial f = 0$ and $\partial(e_1) = \partial(e_2) = 0$. Therefore

$$H_n(\text{torus}) = \begin{cases} \mathbf{Z} & \text{if } n = 0, 2 \\ \mathbf{Z} \oplus \mathbf{Z} & \text{if } n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Homology With Coefficients

If A is an abelian group, we can compute homology with coefficient in A, $H_*(X; A)$, in a similar way, because the only property of $\mathbf{Z}$ we ever used was addition. All we do is work with A-linear combinations of simplexes, singular simplexes, cells, etc. Replace (algebraically) $C_n = \bigoplus_{\text{cellsorsimplexes}} \mathbf{Z}$ by $\oplus A$. Boundary map makes sense over A, $\partial(a \cdot \sigma) = \sum(\pm a)\sigma \ldots$, $a \in A$. Extend A linearly.

Properties All previous definitions work: definition of homology, homotopy invariance, Mayer-Vietoris sequence, excision.

Modifications $H_*(\mathrm{pt};A) \cong \begin{cases} A & \dim 0 \\ 0 & \text{otherwise} \end{cases}$. $H_*(S^{n\geq 1};A) \cong \begin{cases} A & \dim 0, n \\ 0 & \text{otherwise} \end{cases}$. If $\theta : A \to B$ is an abelian group homomorphism, then there exists a map $\theta_* : H_*(X;A) \to H_*(X,B)$ defined in the obvious way algebraically (homomorphism of abelian groups). The degree of a map $f : S^n \to S^n$ can be defined using the homomorphism, for $a \in A$, $\theta_a : \mathbf{Z} \to A$ defined by $1 \mapsto a$, and the commutative diagram

$$\begin{array}{ccc} H_n(S^n;\mathbf{Z}) & \xrightarrow{f_*} & H_n(S^n;\mathbf{Z}) \\ \downarrow \theta_{a*} & & \downarrow \theta_{a*} \\ H_n(S^n;A) & \xrightarrow{\tilde{f}_*} & H_n(S^n;A) \end{array}$$

$\tilde{f}_*$ is then a map taking $a \in A \mapsto da \in H_n(S^n;A)$.

In terms of cellular homology, since the boundary maps can be expressed as degrees of maps $S^{n-1} \to S^{n-1}$ (TO DO: Why?) the effect of goin to A coefficients is to replace $C_n = \mathbf{Z}^m \stackrel{\partial^{\mathrm{cell}}}{\to} \mathbf{Z}^n = C_{n-1}$ (an integer matrix) $\sum_i x_i(\mathrm{cell}_i) \mapsto \sum_{i,j} x_i(\partial_{ij})(\mathrm{cell}_j)$ (where x_i and ∂_{ij} are integers) by a map $A^m \to A^n$, $\sum_i a_i(\mathrm{cell}_i) \mapsto \sum_{i,j} a_i(\partial_{i,j} \in \mathbf{Z})(\mathrm{cell}_j)$.

Example The usual chain complex for $\mathbf{R}P^n$ is:

$$\begin{array}{cccccccc} & n & & n-1 & & 2 & 1 & 0 \\ 0 \to & \mathbf{Z} & \xrightarrow{\times[(-1)^n+1]} & \mathbf{Z} & \to \cdots \to & \mathbf{Z} \xrightarrow{\times 2} & \mathbf{Z} \xrightarrow{\times 0} & \mathbf{Z} \to 0 \end{array}$$

(numbers above $\mathbf{Z}$ indicate dimension). So $H_*(\mathbf{R}P^n;\mathbf{Z})$ is 0 in even dimensions $\leq n$, $\mathbf{Z}_2$ in odd dimensions $< n$, $\mathbf{Z}$ in dimension n if n is odd, and $\mathbf{Z}$ in dimension 0.

$\mathbf{Z}_2$ coefficients Now the chain complex is:

$$0 \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to \cdots \to \mathbf{Z}_2 \to \mathbf{Z}_2 \to 0$$

All boundary maps are $\times\{0 \text{ or } 2\}$, taken mod 2, ie. zero maps. Therefore $H_*(\mathbf{R}P^n;\mathbf{Z}_2)$ is $\mathbf{Z}_2$ in dimensions 0 to n, 0 otherwise.

$\mathbf{Q}$ coefficients

$$0 \to \mathbf{Q} \to \mathbf{Q} \to \cdots \to \mathbf{Q} \to \mathbf{Q} \to \mathbf{Q} \to 0$$

Observe that $\times 2$ is an isomorphism $\mathbf{Q} \to \mathbf{Q}$.

$$H_*(\mathbf{R}P^n;\mathbf{Q}) = \begin{cases} 0 & \text{in dimensions 1 through } n-1 \\ \mathbf{Q} & \text{in dimension 0} \\ 0 & \text{in dimension } n \text{ if } n \text{ even} \\ \mathbf{Q} & \text{in dimension } n \text{ if } n \text{ odd} \end{cases}$$

Aside For $\mathbf{C}P^n$, the chain complex is:

$$\begin{array}{ccccccccc} & 2n & & 2n-1 & & 2n-1 & & 2n-3 & & 2n-4 & & 2 & 1 & 0 \\ 0 \to & \mathbf{Z} & \to & 0 & \to & \mathbf{Z} & \to & 0 & \to & \mathbf{Z} & \cdots & \mathbf{Z} \to & 0 \to & \mathbf{Z} \to 0 \end{array}$$

All boundary maps are (clearly) 0, so $\mathbf{C}P^n$ has homology $\mathbf{Z}$ in even dimensions up to $2n$.

Remark Actually $H_*(X;A)$ is determined by $H_*(X;\mathbf{Z})$ (more than the trivial thing that $C_*(X;\mathbf{Z})$ determines all $H_*(X;\mathbf{Z})$). *Universal Coefficient Theorem.*

Euler Characteristic

Remark fist that with rational coefficients all $H_*(X;\mathbf{Q})$ are $\mathbf{Q}$-vector spaces. Consequently let us define i^{th} *Bette number of* X to be $b_i(X) = \dim H_i(X;\mathbf{Q})$. Let $\chi(X) = \sum_0^\infty (-1)^i b_i(X)$ assuming $b_i = 0$ for large enough i.

Theorem *$\chi(X) = \sum_0^\infty (-1)^i n_i$ where n_i is the number of cells in a cell decomposition of X. Consequently, this sum is an invariant!*

Proof Observe $b_i = \dim H_i = \dim Z_i - \dim B_i$ (defined since Z_i, B_i are $\mathbf{Q}$ vector spaces). Also, $n_i = \dim Z_i + \dim B_{i-1}$ since $C_i/Z_i \cong B_{i-1}$ or $C_i/\ker\partial_i = \operatorname{im}\partial_i$. Then check that the alternationg sums agree:

$$\begin{aligned}
\sum(-1)^i b_i &= \sum(-1)^i \dim Z_i - \sum(-1)^i \dim B_i \\
\sum(-1)^i n_i &= \sum(-1)^i \dim Z_i - \sum(-1)^i \dim B_{i-1} \\
&= \sum(-1)^i \dim Z_i - \sum(-1)^i \dim B_i
\end{aligned}$$

$\surd$

Cohomology

Start with a chain complex $\to C_n \xrightarrow{\partial} C_{n-1} \xrightarrow{\partial} C_{n-2} \to$ (imagine $C_n^{\text{sing}}X$) and dualize it to $C^n = \operatorname{Hom}(C_n,\mathbf{Z})$. WARNING: this allows infinite linear combinations. For an infinitely generated C_n, C^n allows assignments of integers to every singular simplex *without restriction*, ie. if $C_n = \bigoplus_{\alpha\in I}\mathbf{Z} \Longrightarrow C^n = \prod_{\alpha\in I}\mathbf{Z}$.

Definition (Coboundary maps as adjoints of boundary maps) If $A \xrightarrow{\theta} B$ is an abelian group homomorphism, we get $A^* = \operatorname{Hom}(A,\mathbf{Z}) \xleftarrow{\theta^*} \operatorname{Hom}(B,\mathbf{Z})$ by $\theta^*(f) = f\circ\theta$ for $f : B \to \mathbf{Z}$ ($A \xrightarrow{\theta} B \xrightarrow{f} \mathbf{Z}$). Equivalently, θ^* is defined by the property $\langle\theta^*(f), a\rangle = \langle f, \theta(a)\rangle$ where $\langle\ ,\ \rangle$ denotes evaluation. Dualizing the boundary maps gives us coboundary maps $\delta : C^{n-1} \to C^n$ so we have a chain complex $\to C^{n-1} \xrightarrow{\delta} C^n \xrightarrow{\delta} C^{n+1} \to$.

Proof ($\delta^2 = 0$) Let ξ be a functional in C^n and x be in C_{n+2}. Then $\langle\delta^2\xi, x\rangle = \langle\delta\xi, \partial x\rangle = \langle\xi, \partial^2 x\rangle = 0$. $\surd$

Properties All follow from usual singular homology by dualing all the algebra. Though be careful since taking the dual of an exact sequence is not necessarily exact.

(1) For a continuous map $f : X \to Y$, get $f^* : H^*Y \to H^*X$, ie. H^* is a *contravariant functor*. It is *natural* in the sense that $(f\circ g)^* = g^*\circ f^*$ and $\mathrm{id}^* = \mathrm{id}$ where

$$\begin{array}{ccccc}
X & \xrightarrow{f} & Y & \xrightarrow{g} & Z \\
H^*X & \xleftarrow{f^*} & H^*Y & \xleftarrow{g^*} & H^*Z
\end{array}$$

For example, if $[\xi] \in H^*Y$ is a cohomology class (ξ is a cocycle), the cohomology map $f^*[\xi] = [\xi\circ f_*]$ where f_* is a map of chains

$$\begin{array}{ll}
C^nX : & C_nX \to \mathbf{Z} \\
 & \downarrow f_* \\
C^nY : & C_nY \xrightarrow{\xi} \mathbf{Z}
\end{array}$$

(2) Homotopy invariance: If $f \simeq g$ then $f^* = g^*$.

(3) $C_n(A+B) \hookrightarrow C_n(A \cup B)$ was a chain homotopy equivalence $\Longrightarrow$ Mayer-Vietoris sequence for homology. Similarly for the cohomology case, $C^n(A \cup B) \twoheadrightarrow C^n(A+B)$ is a chain homotopy equivalence. Under the usual topological conditions, this gives a Mayer-Vietoris sequence

$$\cdots \to H^n(A \cup B) \to H^n(A) \oplus H^n(B) \to H^n(A \cap B) \xrightarrow{\delta^*} H^{n+1}(A \cup B) \to \cdots$$

The fundamental theorem of hoomological algebra work to give δ^* increasing the dimension by 1.

$$\begin{array}{ccccccc} 0 \leftarrow & \leftarrow & \leftarrow & \leftarrow 0 \\ & \uparrow & \uparrow & \uparrow \\ 0 \leftarrow & \leftarrow & \leftarrow & \leftarrow 0 \end{array}$$

(4) Relative cohomology can be defined as usual for (X, A) via the exact sequence

$$0 \to C^n(X, A) \to C^n(X) \overset{i^*}{\twoheadrightarrow} C^n(A) \to 0$$

Where $C^n(X, A)$ is defined to be the kernel of restriction (free and spanned by assignments to singular simplexes not in A), and $i : A \hookrightarrow X$ so $i^* : C^n X \twoheadrightarrow C^n A$. This gives a l.e.s. of relative cohomology containing

$$H^{n-1}(A) \to H^n(X, A) \to H^n(X) \to H^n(A) \xrightarrow{\delta} H^{n+1}(X, A)$$

(5) Excision as before. If we have the closure of U contained in the interior of A then restriction gives $\cong H^*(X, A) \to H^*(X - U, A - U)$.

Cellular cochain complex can be used to compute cohomology groups. From this we see eg.

$$H^*(\mathrm{pt}) \cong \begin{cases} \mathbf{Z} & \text{dim } 0 \\ 0 & \text{otherwise} \end{cases}$$

(cochain is $0 \to \mathbf{Z} \to 0$).

$$H^*(S^n) \cong \begin{cases} \mathbf{Z} & \text{dim } 0, n \\ 0 & \text{otherwise} \end{cases}$$

Observe there are examples where they differ (abstract chain complexes):

$$\begin{array}{ccc} 0 \to \mathbf{Z} \overset{\times 2}{\to} \mathbf{Z} \to 0 & \text{dual} & 0 \leftarrow \mathbf{Z} \overset{\times 2}{\leftarrow} \mathbf{Z} \leftarrow 0 \\ H_0 = \mathbf{Z}_2,\ H_1 = 0 & & H^0 = 0,\ H^1 = \mathbf{Z}_2 \end{array}$$

eg. $\mathbf{R}P^2$:

$$0 \to \underset{2\text{-cell}}{\mathbf{Z}} \overset{\times 2}{\to} \underset{\text{edge}}{\mathbf{Z}} \overset{0}{\to} \underset{\text{point}}{\mathbf{Z}} \to 0$$

$H_0 \cong \mathbf{Z}$, $H_1 \cong \mathbf{Z}_2$, $H_2 \cong 0$, $H^0 = \mathbf{Z}$, $H^1 = 0$, $H^2 = \mathbf{Z}_2$. Also consider a bouquet of I-indexed circles

$$0 \to \bigoplus_{\alpha \in I} \mathbf{Z} \to 0 \quad 0 \leftarrow \prod \mathbf{Z} \leftarrow 0$$

where $H_1 = \bigoplus \mathbf{Z}$ and $H^1 = \prod \mathbf{Z}$.

Definition *Kronecker pairing* is the map $H^n(X) \times H_n(X) \to \mathbf{Z}$ induced by evaluation $\langle , \rangle : C^n(X) \times C_n(X) \to \mathbf{Z}$ by $\langle \xi, \sigma \rangle = \xi(\sigma)$ (bilinear). This is independent of representative cocycle choice since given cocycle ζ and cycle z, then $\langle \zeta + \delta\xi, z + \partial x \rangle = \langle \zeta, z \rangle + \langle \zeta, \partial x \rangle + \langle \delta\xi, z \rangle + \langle \delta\xi, \partial x \rangle = \langle \zeta, z \rangle + \langle \delta\zeta, x \rangle + \langle \xi, \partial z \rangle + \langle \xi, \partial^2 x \rangle = \langle \zeta, z \rangle$. Therefore we may view this as a map $H^n X \to \mathrm{Hom}(H_n X, \mathbf{Z})$. In general this is *not* an isomorphism. However if we work with *field* coefficients, all chain groups (etc.) are vector spaces, and many things simplify. In particular Kronecker pairing is an isomorphism. Proof on sheet. (TO DO: find sheet)

A picture of simplicial cohomology of the torus We know $H_* T^2$ is generated by a point (p), two 1-cycles (a, b), and $\sum$all 2-simplexes (f). Picture *cochains* in the same way: label an edge by an arrow to denote the cochain which sends it (with that orientation) to $1 \in \mathbf{Z}$. On cochains $\langle \delta(\text{copt} \in C^0), \nearrow \text{edge} \in C_1 \rangle = \begin{cases} +1 & \text{if point at end of edge} \\ -1 & \text{if at start of edge} \\ 0 & \text{otherwise} \end{cases}$.
So $\delta(\text{copt}) = \sum$ of all coedges ending at that point $(= -\sum$ coedges leaving that point). Similarly

$$\langle \delta(\nearrow), \text{oriented } 2-\text{simplex} \rangle = \begin{cases} \pm 1 & \text{if the edge is a face of the 2-simplex} \\ & \text{(negative if orientations disagree)} \\ 0 & \text{otherwise} \end{cases}$$

So $\delta(\nearrow)$ is the sum of the two cofaces which share that edge. Finally, the coboundary of a 2-simplex is zero.

So use $\delta(\nearrow)$ to push all "charge" onto a 2-cocycle (a 2 co-chain) to a single face. Claim: this is a generator for H^2 since total charge is conserved. Therefore $H^2(T^2) \cong \mathbf{Z}$, generated by one chosen "coface." Very much like H_0, think of H_0 on the dual figure with vertices exchanged with faces, etc. Also $H^0(torus) \cong \mathbf{Z}$ generated by $\sum$(copts) in diagram. The procedure mirrors the computation of H_2. $H^1(T^2)$ is generated by α, β where α consists of all the $\uparrow$ and $\nearrow$ edges in one row of the complex and β consists of all the $\rightarrow$ and $\nearrow$ edges in one column of the complex. Observe that the evaluation $\langle , \rangle : H^1 \times H_1 \to \mathbf{Z}$ is given by $\langle \alpha, a \rangle = +1$, $\langle \alpha, b \rangle = 0$, $\langle \beta, a \rangle = 0$, and $\langle \beta, b \rangle = +1$. Consequently, Kronecker pairing *is* an isomorphism here.

Cup Product

Fix a commutative ring R with a unit 1. Work with $H^*(X; R)$ — since R-linear combinations of simplexes can be multiplied by elements of R, the result is an R-module (all homomorphisms are R-module homomorphisms). We can get a *product* on cohomology classes (and not "naturally" on homology). The moral reason is that (1) $H^*(X \times X)$ is, more or less, $H^*X \otimes H^*X$ (true for fields); and (2) any space has a diagonal map $\Delta : X \to X \times X$ which is continuous, so we can map $H^*X \otimes H^*X \xrightarrow{\simeq} H^*(X \times X) \xrightarrow{\Delta^*} H^*(X)$ to define a product.

Definition In singular homology, suppose $\zeta \in C^m(X), \xi \in C^n(X)$. Define *front* and *back face maps* $\alpha_m : \Delta^m \to \Delta^{m+n}$ $(e_0, \ldots, e_m, 0, \ldots, 0)$ $\beta_n : \Delta^n \to \Delta^{m+n}$ $(0, \ldots, 0, e_0, \ldots, e_n)$ (note $\alpha_m(e_m)$ overlaps $\beta_n(e_0)$). Now define $\zeta \cup \xi$ on $(m+n)$-cochain by $\langle zeta \cup \xi, \sigma : \Delta^{m+n} \to X \rangle = \langle \zeta, \sigma \circ \alpha_m \rangle \cdot_R \langle \xi, \sigma \circ \beta_n \rangle$ and extend linearly.

Properties (1) $\delta(\zeta \cup \xi) = \delta\zeta \cup \xi + (-1)^m \zeta \cup \delta\xi$.

Proof Use the defintion $\langle \delta(\zeta \cup \xi), \sigma \rangle = \langle \zeta \cup \xi, \partial\sigma \rangle$. $\surd$

(2) There exist induced maps $(H^m X) \times (H^n X) \xrightarrow{\cup} H^{m+n} X$.

Proof If ζ, ξ are *cocycles* then $\zeta \cup \xi$ is a cocycle. If we take a product of a coboundary $\delta\zeta$ with a cocycle ξ we get a coboundary $(\delta\zeta) \cup \xi = \delta(\zeta \cup \xi)$. These are immediate from property (1). $\surd$

(3) It is *natural*: Given $f : X \to Y$, we have $f^*([\zeta] \cup [\xi]) = f^*[\zeta] \cup f^*[\xi]$ for $[\zeta], [\xi] \in H^*Y$.

$$\begin{array}{ccccc} H^m X & \times & H^n X & \to & H^{m+n} X \\ \uparrow f^* & & \uparrow f^* & & \uparrow f^* \\ H^m Y & \times & H^n Y & \to & H^{m+n} Y \end{array}$$

(4) It is R-bilinear, associative, and there is an identity element

$$1 \in H^0 X = [\text{cocycle}: \text{ everything} \mapsto 1 \in R] .$$

(Proof: just check).

(5) Graded Commutativity: $[\zeta] \cup [\xi] = (-1)^{mn} [\xi] \cup [\zeta]$ where $[\zeta] \in H^m X, [\xi] \in H^n X$.

Proof On a handout. Remark: this is *not* true at the level of cocycles. $\surd$

(6) Conclude that $H^* X = \bigoplus_{m \geq 0} H^m X$ ($\bigoplus$ as R-modules) is a *graded ring*. Ie. it is an R-module, with multiplication, unit, grading, associativity, "skew graded-commutativity." It is a contravariant functor from spaces to graded rings (in particular, f^* is a ring homomorphism) which preserves grading.

To compute using simplicial homology: order all the vertices in a complex (so 'front' and 'back' make sense for each simplex). Then use the same formula as for singular homology. The answer is independent of the ordering of vertices chosen.

Eg. on torus

$$H^*(T; \mathbf{Z}) \simeq \underset{H^0}{\mathbf{Z}} \oplus \underbrace{\mathbf{Z}_\alpha \oplus \mathbf{Z}_\beta}_{H^1} \oplus \underset{H^2}{\mathbf{Z}_\sigma}$$

H^0 is identified with "scalar" $\mathbf{Z}$. $\langle \alpha \cup \beta, \text{a 2-simplex} \rangle = 0$ for all but one 2-simplex. Therefore can write $H^* T \cong \bigwedge^*(\alpha, \beta)$ where α, β are elements of degree 1. So degree $0 = \mathbf{Z}$, $1 = \mathbf{Z}_\alpha \oplus \mathbf{Z}_\beta$, $2 = \mathbf{Z}(\alpha \cup \beta = \alpha \wedge \beta)$, and ≥ 3 is zero.

Remark 1 Can do a relative product too: use the same definition to induce a map $H^*(X, A) \times H^*(X) \to H^*(X, A)$.

Remark 2 There is an adjoin operation called *cap product* (See handout).

Remark 3 Cup product information is *not* contained in cellular chain complex. Eg. the cellular chains of the torus $0 \to \mathbf{Z} \xrightarrow{0} \mathbf{Z} \oplus \mathbf{Z} \xrightarrow{0} \mathbf{Z} \to 0$.

Orientation and Poincaré Duality for Manifolds

Definition An *n-manifold* M is a Hausdorff topological space with every point having a neighborhood homeomorphic to $\mathbf{R}^n$.

Definition An *n-manifold with boundary* allows some points to have neighborhoods homeomorphic to $\mathbf{R}^n_+$ = closed upper half space.

Remark Note that we can consider the *local homology* of a point x in M, defined as $H_n(M, M - \{x\}) \cong H_n(U, U - \{x\})$ for some neighborhood U of x. Observe (using an exact sequence, etc.) $H_n(\mathbf{R}^n, \mathbf{R}^n - \{0\}) \cong \mathbf{Z}$ and $H_n(\mathbf{R}^n_+, \mathbf{R}^n_+ - \{0\}) \cong 0$. This distinguishes whether x's neighborhoods are $\mathbf{R}^n_+$ or $\mathbf{R}^n$. Consequently points have a well-defined type of neighborhood, and $\partial M = \{x \in M \text{ s.t. neighborhood} \cong \mathbf{R}^n_+\}$ is a well-defined subset. In fact ∂m is an $(n-1)$-manifold with no boundary — $\partial^2 = 0$.

Remark Since $\partial(\partial M) = 0$, we can form a chain complex $C^n = \{$ compact n-manifolds, possibly with boundary $\}$, with ∂ as the boundary operator, and $\mathbf{Z}_2$ coefficients. The homology H_* of this this chain complex is called *unoriented cobordism*, $\Gamma_* = \mathbf{Z}_2\ 0\ \mathbf{Z}_2(\text{from } \mathbf{R}P^2)\ldots$.

Definition Defin the *dual* to a complex by replacing each face by a vertex at the center, each edge with a transverse edge (now connecting the center of two faces instead of two vertices), and each vertex with a face (with one side for every edge that used to be incident). Notice how the dual is topologically like the original. The result is not simplicial, but polyhedral which is nice enough. Given a homology chain complex

$$0 \to C_2 \to C_1 \to C_0 \to 0$$

dualizing gives the homology chain complex of the dual

$$0 \to D_2 \to D_1 \to D_0 \to 0$$

which has some corresopondence to the cohomology chain complex

$$0 \to C_0^* \to C_1^* \to C_2^* \to 0$$

In n-manifolds swap k and $n-k$ dimensional transverse objects. However we don't know how to find cell decompositions—simplicial cell decompositions don't necessarily exist, CW complexes exist but this is hard to show.

Orientation

(In smooth manifolds you may decide on an orientation by choosing a basis fro the tangent space.) If M is a manifold, an orientation at $x \in M$ is a choice of generator of $H_n(M, M-x)$. If U is an open ball neighborhood of x, then by excision $H_n(M, M-x) \cong H_n(U, U-x) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - x) \overset{\text{l.e.s.}}{\cong} H_{n-1}(\mathbf{R}^n - x) \overset{\text{deformation retract}}{\cong} H_{n-1}(S^{n-1}) \cong \mathbf{Z}$. If we are working over an arbitrary ring R, we choose a unit in $H_n(M, M-x; R) \cong R$.

Remark For $\mathbf{Z}_2$, there is a *unique* local orientation.

Definition M^n is *orientable* if there exists a global assignment of generators $x \to \mu_x \in H_n(M, M-x)$ satisfying: For any point x, there exists a compact neighborhood N and a class $\mu_N \in H_n(M, M-N)$ such that for all $y \in N$, $\mu_y = \rho_y(\mu_N)$ where ρ_y is "restriction" (not to be confused with the restriction from cohomology) of (the identity map of geometry) $i_* : H_n(M, M-N) \to H_n(M, M-y)$.

Theorem *If M^n is orientable (in the above sense), for* any *compact set $K \subseteq M$, there exists a class $\mu_K \in H_n(M, M-K)$ restricting to $\mu_x \forall x \in K$. In particular, if M itself is compact, take $K = M$; there exists a* fundamental *orientation* class *$\mu_M \in H_n(M)$, restricting to all μ_x. These are* unique *— it is an analogue of "$\sum$ all n-simplexes coherently oriented."*

Lemma *If K is compact in M, (1) $H_i(M, M-K) = 0 \forall i > n$ (2) if $a \in H_n(M, M-K)$ and $\rho_x(a) = 0 \forall x \in K$, then $a = 0$ (sheaf property).*

Proof (Lemma) Basic idea is if it is true for compact sets $K_1, K_2, K_1 \cap K_2$, then true for $K_1 \cup K_2$. We use Mayer-Vietoris sequence for pairs $(M, M - K_1)$ and $(M, M - K_2)$

$$H_{i+1}(M, M - K_1 \cap K_2) \to H_i(M, M - K_1 \cup K_2) \to H_i(M, M - K_1) \oplus H_i(M, M - K_2) \\ \to H_i(M, M - K_1 \cap K_2) \to \cdots$$

(1) immediately follows since $H_i(M, M - K_1 \cup K_2)$ is "sandwiched between '0's." To get (2) observe that the leftmost group is zero when $i = n$. So $H_n(M, M - K_1 \cup K_2)$ injects into $H_i(M, M - K_1) \oplus H_i(M, M - K_2)$ using $i_{1*} \oplus -i_{2*}$, or equivalently "restricts" to $H_i(M, M - K_1) \oplus H_i(M, M - K_2)$. Since restriction $\forall x \in K_1 \cup K_2$ is zero by hypothesis, we conclude $(i_{1*} \oplus -i_{2*})(a) = 0 \Longrightarrow a = 0$ by injectivity.

Now use this glueing part to prove the Lemma in stages: It is true for any convex compact set K in $\mathbf{R}^n$ since

$$\begin{array}{ccccl} H_i(\mathbf{R}^n, \mathbf{R}^n - K) & \overset{\text{l.e.s.}}{\cong} & H_{i-1}(\mathbf{R}^n - K) & \cong H_{i-1}(S^n - 1) & \text{by a deformation retract} \\ & {\scriptstyle \rho_x} \searrow \simeq & & \nearrow \simeq & \\ & & H_i(\mathbf{R}^n, \mathbf{R}^n - x) & & \end{array}$$

So there exists only one choice of μ_K here. This shows the result is true for a finite simplicial complex in $\mathbf{R}^n$ using the above and gluing. Look in Hatcher for proof for *any* compact K in $\mathbf{R}^n$. Sketch: For $a \in H_i(\mathbf{R}^n, \mathbf{R}^n - K)$, let z be a (relative) cycle in $C_i(\mathbf{R}^n)$ representing it. The image of ∂z is a compact set C not meeting K, so there is some positive distance between them (using compactness since C is a *finite* linear combination of edges, uses Hausdorff property of manifold). Now subdivide all of $\mathbf{R}^n$ into tiny simplexes (smaller than the distance between C and K); replace K by N, the union of all the tiny simplexes meeting K. Note that N still misses C. z represents an element $a_N \in H_i(\mathbf{R}^n, \mathbf{R}^n - N)$ which restricts $\rho_K(a_N) = a \in H_i(\mathbf{R}^n, \mathbf{R}^n - K)$. (1) If $i > n$, $a_N = 0$ (we know Lemma for N), and $a = \rho_K(a_N) = 0$. Good. (2) If $i = n$, observe $\rho_x(a_N) = 0 \forall x \in K$ (since $\rho_x(a_N) = \rho_x \rho_K(a_N) = \rho_x(a) = 0$ by assumption). Also, if $x \in$ a simplex near the boundary ($x \notin k$), there exists another y in a simplex inside K with $\rho_y(a_N) = 0$, consequently since $H_n(\mathbf{R}^n, \mathbf{R}^n - x) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - \sigma) \cong H_n(\mathbf{R}^n, \mathbf{R}^n - y)$. Conclude $\rho_y(a_N) = 0$ too. Hence $a_N = 0$ and $a = 0$.

This shows the Lemma is true for any $K \subseteq M$, by considering K to be a union of finitely many compact sets, each contained in a coordinate neighborhood, for which (by the previous bit) the Lemma is true. $\surd$

Editor's Note 'glueing' and 'gluing' are both valid spellings. Decide on one.

Proof (Theorem) Uniqueness from the Lemma construction by the same glueing procedure:

$$0 \to H_n(M, M - K_1 \cup K_2) \overset{i_{1*} \oplus -i_{2*}}{\longrightarrow} H_n(M, M - K) \oplus H_n(M, M - K_2) \overset{j_{1*} + j_{2*}}{\longrightarrow} H_n(M, M - K_1 \cap K_2)$$

Since $j_{1*} + j_{2*}(\mu_{K1} \oplus -\mu_{K2}) = \mu_{K_1 \cap K_2} - \mu_{K_1 \cap K_2}$, it comes from some $\mu_{K_1 \cup K_2}$. The rest of the proof is the same as before. $\surd$

Corollary *A compact orientable manifold has $H_n(M; \mathbf{Z}) \neq 0$ (locally its restriction is a* generator $\neq 0$*).*

Definition *Any* connected manifold M has an *orientation (double) cover* $\tilde{M}$, defined as follows: $\tilde{M} = \{(x, \mu_x) : \forall x \in M, \mu_x \text{ generator of integer local homology}\}$. It is a manifold with coordinate balls $\{(x, \mu_x) : x \in B \subseteq M, \mu_x \text{ is restriction of } \mu_B\}$ where B is a coordinate ball, and some fiddely bits have been glossed over regarding μ_B. Also, these coordinate balls generate the topology on the double cover. It is a double cover since locally homoemorphic an there exist two preimages $(x, \pm\mu_x)$ for x. Consequently we get an *orientation homomorphism* $w : \pi_1 M \to \mathbf{Z}_2$ defined as monodromy of $\tilde{M}$.

Lemma *For connected manifolds M, M is orientable $\iff$ $\tilde{M}$ is disconnected (two components).*

Proof Check that $\tilde{M}$ is always orientable; so if it has two components, each is homeomorphic to M, and therefore M is orientable. Conversely, if $\tilde{M}$ is connected there exists a path which shows M not oriented. *Equivalently* M is orientable $\iff$ homomorphism w is trivial. $\surd$

Corollary *Any triple cover of a Klein bottle is non-orientable.*

Proof If $\hat{K}$ is a 3-cover, $\pi_1 \hat{K} \overset{p_*}{\hookrightarrow} \pi_1 K \overset{w}{\to} \mathbf{Z}_2$. This composition is the monodromy $\hat{w}$ for $\hat{K}$. So since $p_* \pi_1 \hat{K}$ is index 3, it cannot be $\leq$ the index 2 kernel of w. Therefore $\hat{K}$ is not orientable. $\surd$

Definition A cochain in $C^i(M)$ has *compact support* if there exists a compact K such that the cochain vanishes on all singular simplexes contained in $M - K$. In other words, since $0 \to C^i(M, M-K) \to C^i(M) \to C^i(M-K) \to 0$, compactly supported cochains are ones in the subgroup $C^n(M, M-K)$ for some K. Use $\langle \delta f, x\rangle = \langle f, \partial x\rangle$ to show δ takes compactly supported chains to compactly supported chains, hence get the cochain complex $H_c^*(M)$.

Definition (Alternative) A *directed set* is an indexing set A with a partial order $\leq$, such that given any two elements $\alpha, \beta \in A$, there exists γ bigger than both, so $\alpha \leq \gamma, \beta \leq \gamma$. A *direct system* of abelian groups is a family $\{G_\alpha : \alpha \in A\}$ with A a directed set such that for all $\alpha \leq \beta$ there exists a homomorphism $G_\alpha \overset{f_{\alpha\beta}}{\to} G_\beta$ satisfying $f_{\alpha\beta} \circ f_{\beta\gamma} = f_{\alpha\gamma}$ whenever $\alpha \leq \beta \leq \gamma$. The *direct limit* of $\{G_\alpha\}$ is just $\varinjlim G_\alpha = \bigoplus_{\alpha \in A} G_\alpha / \{x - f_{\alpha\beta}(x) : \forall x \in G_\alpha, \beta \geq \alpha\}$, an abelian group. So we may rephrase the definition of compactly supported cohomology $H_c^*(M)$ using the directed set of all compact subsets of M ordered by inclusion, the direct system $\{H^i(M, M-K)\}$ with morphisms $H^j(M, M-K) \overset{i_*}{\to} H^j(M, M-L)$ where i_* is "restriction."

Editor's Note In the following H^1 should probably be H^i but the following is what is in my notes. Probably a transcription error.

Lemma $\varinjlim H^1(M, M-K) \cong H_c^1(M)$.

Proof Show injective and surjective. Proof is just like the cell complex homology proof. $\surd$

Lemma *Direct limits preserve exactness. Ie. if $0 \to A_\alpha \to B_\alpha \to C_\alpha \to 0$ for all $\alpha \in I$ a directed set with $\{A_\alpha\}, \{B_\alpha\}, \{C_\alpha\}$ directed systems, then $0 \to \varinjlim A_\alpha \to \varinjlim B_\alpha \to \varinjlim C_\alpha \to 0$ is still exact. (Look up the proof in some book)*

Poincaré Duality *Let M^n be a manifold (with* no *boundary). If M is orientable, we work with integer coefficients (or any others!); if non-orientable, use $\mathbf{Z}_2$ (or characteristic 2). ("The integers upside down" ?) So we have fundamental classes $\mu_K \in H_n(M, M-K)$. Then "cap product with μ" gives an isomorphism $H_c^i(M) \overset{\text{"}\cap\mu\text{"}}{\longrightarrow} H_{n-i}(M)$. In particular, if M is compact, $H^i(M) \overset{\cap\mu}{\longrightarrow} H_{n-i}(M)$.*

Proof Stage i) First lets define the operation "$\cap\mu$." Given an element of $H_c^i(M)$

represent it by $\xi_K \in H^i(M, M-K)$, for some K. Then cap with μ_K to get something in $H_{n-i}(M)$ using relative cap product. If we choose a different representative ξ_L, with L compact $K \subseteq L$, then

$$\begin{array}{ccc} \xi_K \in H^i(M, M-K) \times \mu_K \in H_n(M, M-K) & \xrightarrow{\cap} & H_{n-1}(M) \\ i^* \downarrow \qquad\qquad\qquad \uparrow i_* & & \| \\ \xi_L \in H^i(M, M-L) \times \mu_L \in H_n(M, M-L) & \xrightarrow{\cap} & H_{n-1}(M) \end{array}$$

commutes by naturality of $\cap$. Since $i^*(\xi_K) = \xi_L$ and $i_*(\mu_L) = \mu_K$ (by uniqueness), we get the same answer for δK as for $L \supseteq K$. Consequently it is the same for K_1, $K_1 \cup K_2$, K_2 for arbitrary K_1, K_2, and the operation "$\cap\mu$" is well-defined.

Stage ii) Prove if theorem is true for open sets $U, V, U \cap V$ then true for $M = U \cup V$. Suppose $K \subseteq U, L \subseteq V$ compact.

$$\begin{array}{ccccccc} H^i(M, M-(K \cap L)) & \to & H^i(M, M-K) \oplus H^i(M, M-L) & \to & H^i(M, M-(K \cup L)) & \xrightarrow{\delta} \\ \cong \downarrow \text{excision} & & \cong\downarrow \text{excision} \qquad \cong \downarrow \text{excision} & \nearrow & \Big| & \searrow \\ H^i(U \cap V, (U \cap V) - (K \cap L)) & & H^i(U, U-K) \oplus H^i(V, V-L) & & \Big|\cap\mu_{K \cup L} & \\ \downarrow \cap\mu_{K \cap L} & & \downarrow \cap\mu_K \qquad\qquad \downarrow \cap\mu_L & & \downarrow & \\ H_{n-i}(U \cap V) & \to & H_{n-i}(U) \oplus H_{n-i}(V) & \to & H_{n-i}(M) & \xrightarrow{\delta} \end{array}$$

(Imagine two more columns of maps on the right, headed by $H^{i+1}(M, M-(K \cap L))$ and $H^{i+1}(M, M-K) \oplus H^{i+1}(M, M-L)$) *IF* this all commutes, we take the direct limit (preserving exactness), get four (outer) vertical isomorphisms, and then apply the 5-lemma to get the middle vertical isomorphism.

Stage iii) Also need "infinite" gluing procedure: If theorem is true for each U_j, $U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$ LOOK UP the rest in Hatcher's Notes. $\surd$

Corollary *Assume M is compact, connected, closed manifold. Then M is orientable $\iff H_n(M; \mathbf{Z}) \cong \mathbf{Z}$ (nonorientable $\iff H_n(M; \mathbf{Z}) \cong 0$).*

Proof ($\Longrightarrow$) First $H^i_c(M) = H^i(M)$ since M is compact. By Poincaré duality, $H^i(M) \cong H_{n-i}(M)$. Since M is connected, $H^0 \cong \mathbf{Z} \cong H_n$. On the other hand, if $H_n(M) \neq 0$; pick an element $\mu_m \in H_n$. It restricts at some point x to $q \cdot \mu_x \in H_n(M, M-x)$ for some generator μ_x, $q \in \mathbf{Z}$. Take $1/q$ times this class — it generates, so M is orientable. $\surd$

Lefshetz (sp?) duality *If M^n is a compact manifold with boundary ∂M, it is orientable if $M - \partial M$ is orientable. Then there exists a fundamental class $\mu_{M,\partial M} \in H_n(M, \partial M) \cong \mathbf{Z}$.*

Proof Same idea as Poincaré duality. Note $\cap\mu_{M,\partial M} : H^i(M) \xrightarrow{\cong} H_{n-i}(M, \partial M)$ is an *isomorphism.* $\surd$

Remark 1 For $i = n$, $\partial_*(\mu_{M,\partial M}) = \mu_{\partial M}$:

$$\to H_i(\partial M) \to H_i(M) \to \mu_{M,\partial M} \in H_i(M, \partial M) \xrightarrow{\partial_*} \mu_{\partial M} \in H_{i-1}(\partial M) \to$$

Remark 2 Lefschetz duality is "natural" with respect to the l.e.s. of homology and cohomology.

Alexander Duality *If $S^n = A \cup B$ is a decomposition of S^n into two with boundary ($A \cap B = \partial A = \partial B$) (eg: knotted solid torus and its complement in S^3). Then $\tilde{H}_{i-1}(A) \cong \tilde{H}^{n-i}(B)$.*

Proof Note that this is trivial for $i < 1$ or $i > n$. Otherwise,

$$\tilde{H}_i(S^n) \to \tilde{H}_i(S^n, A) = H_i(S^n, A) \overset{\text{excision}}{\cong} H_i(B, A \cap B = \partial B) \overset{\text{Lefschetz}}{\cong} \tilde{H}^{n-1}(B)$$
$$\to \tilde{H}_{i-1}(A) \to \tilde{H}_{i-1}(S^n) \to$$

For $i \neq n$, end two are zero, therfore the middle is an isomorphism. (CHECK for something missing here) $\surd$

Representations by submanifolds

Assume everything is a smooth manifold.

Definition If $N^n \subseteq M^m$ is a closed (closed means compact and without boundary) oriented submanifold of some manifold M, it *represents* a homology class via the map i_* : $H_n(N) \to H_n(M)$ $\mu_N \mapsto i_*(\mu_N) = [N]$. Eg. $H_*(T^2)$ is generated by a point, two circles, and the whole torus.

Definition Two submanifolds $N_1^{n_1}, N_2^{n_2}$ of M^m are *transverse* if their intersection $N_1 \cap N_2$ is an $n_1 + n_2 - m$ dimensional submanifold, and at each point $x \in N_1 \cap N_2$, there exists a diffeomorphism for some ball neighborhood of x in M to $\mathbf{R}^m$, taking $(B^m, B^m \cap N_1^{n_1}, B^m \cap N_2^{n_2}) \to (\mathbf{R}^m, \mathbf{R}^{n_1} \times \{0\}^{m-n_1}, \{0\}^{m-n_2} \times \mathbf{R}^{n_2})$. The most useful case is where $n_1 + n_2 = m$, and the intersection is a discrete set of points.

Fact In a smooth manifold, two nontransverse submanifolds may be perturbed (by ϵ) to make them transverse.

Definition If $[N_1], [N_2]$ are two homology classes (made transverse), their *geometric intersection* $[N_1] \cdot [N_2]$ is the homology class of $[N_1 \cap N_2]$.

Remark 1 Everything is *oriented.* Each tangent space TM_x (over $H_n(M, M - x) = H_n(B^n, B^n - x)$) is oriented (by choosing a basis and work modulo changes of basis with positive determinant). Also $(TN_1)_x, (TN_2)_x$ are oriented. Then we give $T(N_1 \cap N_2)_x$ an orientation induced from these three.

$$\begin{array}{ccccc} 0 \to T(N_1 \cap N_2) \to & (TN_1)_x & \oplus & (TN_2)_x & \to TM_x \to 0 \\ \text{basis vectors}: \{e, \ldots, e\} & \{f, \ldots, f, e, \ldots, e\} & & \{e, \ldots, e, g, \ldots, g\} & \{f; e; g\} \end{array}$$

In this way crossing $+1$ or -1 can be made consistent. (This allows a path to cross a line from left to right to be given the same *signed* number of intersections as a path which crosses left to right, back to left, and again to the right). If $n_1 + n_2 = m$ all we check is whether (basis of N_1) $\oplus$ (basis of N_2) agrees with (basis for M).

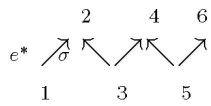
Remark 2 $[N_1] \cdot [N_2] = (-1)^{n_1 n_2} [N_2] \cdot [N_1]$.

Definition *Algebraic intersection*

$$\begin{array}{ccc} H_{n_1}(M) \times H_{n_2}(M) & \overset{p\text{ duel}}{\longrightarrow} & H^{m-n_1}(M) \times H^{m-n_2}(M) \\ \downarrow \text{algebraic intersection} & & \downarrow \cup \\ H_{n_1+n_2-m}(M) & \overset{\text{unduel}}{\longleftarrow} & H^{2m-n_1-n_2}(M) \end{array}$$

Theorem *(See Bredon "Geomtery and Topology") The geometric intersection $[N_1] \cdot [N_2]$ agrees with the algebraic intersection.*

Torus $H_*(T^2)$ recall is generated by a point p, two paths a, b, and $\mu =$"$\sum\sigma$" the sum of all the 2-simplexes coherently oriented (thinking simplicially). Similarly $H^*(T^2)$ is generated by $\sum p^*$ the function returning the sum of all the vertices, zig-zagging paths α, β, and σ^* the function returning the charge on a single two simplex. Already know Kronecker duality $\langle \alpha, a\rangle = \langle \beta, b\rangle = +1$ $\langle \text{other}\rangle = 0$. Use this to work out Poincaré Duals, eg. $D(\alpha) = \alpha \cap \mu \in H_1(T^2)$. We may compute this geometrically by ordering the vertices arbitrarily and using the front/back face definitions.



$(12)^* \cdot e^* \cap \sigma = \langle e^*, 12\rangle \cdot (23) = +1 \times (23)$; $(23)^* \cap \sigma = 0$; Keep doing this, get something that looks like b, so $D(\alpha) = \alpha \cap \mu = b$. Alternatively $\langle \xi, \alpha \cap \mu\rangle = \langle \xi \cup \alpha, \mu\rangle$ or work this out $\alpha \cup \alpha = 0$, $\beta \cup \alpha = \pm\sigma^*$.

Two formula $\Longrightarrow$using Kroneker duality $\alpha \cap \mu = b$.

$$[N_1] \cdot [N_2] = \mathrm{Dual}(\mathrm{Dual}[N_1] \cup \mathrm{Dual}[N_2])$$

$$\langle D[N_1], [N_2]\rangle = [N_1] \cdot [N_2]$$

The second of which is especially useful. The dual of a is a cohomology class measuring how many times things hit a (transversely).

Moral Think of cohomology classes by their dual, which are represented (where possible) by submanifolds.

Example Consider $S^2 \times S^4$.

$$\left.\begin{array}{rcccc} & 0 & 2 & 4 & 6 \\ H_* = & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \\ H^* = & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} & \mathbf{Z} \end{array}\right\}\text{cell complex}$$

The homology has representitives pt $\times$ pt, $S^2 \times$ pt, pt $\times S^4$, $S^2 \times S^4$. Intersections here are clear:

$$\begin{aligned} (S^2 \times \mathrm{pt}) \cap (S^2 \times \mathrm{pt}) &= \emptyset \\ (S^2 \times \mathrm{pt}) \cap (\mathrm{pt} \times S^4) &= \mathrm{pt} \times \mathrm{pt} \\ (\mathrm{pt} \times S^4) \cap (\mathrm{pt} \times S^4) &= \emptyset \end{aligned}$$

Can find cohomology ring of this thing. Take generators $\sum p^*$, $\alpha \in H^2$ dual to pt $\times S^4$, $\beta \in H^4$ dual to $S^2 \times$ pt, and $\sigma^* \in H^6$. Know $\alpha \cup \alpha = 0 = \beta \cup \beta$, and $\alpha \cup \beta = +1 = \beta \cup \alpha$, since $\mathrm{Dual}(\alpha \cup \alpha) = (\mathrm{pt} \times S^4) \cap (\mathrm{pt} \times S^4) = 0$, and similarly for the others.

Example Consider $\mathbf{C}P^3$ which has cohomology $H^0 \cong \mathbf{Z}$ dual to $\mathbf{C}P^3$, $H^2 \cong \mathbf{Z}$ dual to $\mathbf{C}P^2$ (say generated by ξ), $H^4 \cong \mathbf{Z}$ dual to $\mathbf{C}P^1$, and $H^6 \cong \mathbf{Z}$ dual to a point. However the ring structure differs from the cohomology of $S^2 \times S^4$, in particular $\xi \cup \xi \neq 0$ since $\mathbf{C}P^2 \cap \mathbf{C}P^2 = \mathbf{C}P^1$. Therefore $S^2 \times S^4 \not\cong \mathbf{C}P^3$.