Differentiable Manifolds Study Guide

Josh Levenberg

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Manifolds

Definition *Manifold*

Theorem *If $U \subset \mathbf{R}^n$ is open and $f : U \to \mathbf{R}^n$ is one-to-one and continuous, then $f(U) \subset \mathbf{R}^n$ is open. (It follows that $f(V)$ is open for any open $V \subset U$ so f^{-1} is continuous, and f is a homeomorphism.)*

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Theorem *If X is a connected, locally compact metric space, then X is σ-compact.*

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Example of manifolds: S^1 with charts via stereographic projection.

Definition A *surface* is a compact 2-manifold.

Example $M_1 \times M_2$ is a manifold if M_1, M_2 are. So $T^n = (S^1)^n$ the n-torus is a manifold.

Example Mobius strip, Real projective plane = a mobius strip with a disc attached to the boundary. Klein bottle.

Definition *Manifold with boundary*

Differentiable Structures

Definition If U and V are open in M, and $x : U \to x(U) \subset \mathbf{R}^n$, $y : V \to y(v) \subset \mathbf{R}^n$, then x and y are said to be *C^∞-related* if the maps

$$y \circ x^{-1} : x(U \cap V) \to y(U \cap V)$$
$$x \circ y^{-1} : y(U \cap V) \to x(U \cap V)$$

are C^∞.

Definition *C^∞ Manifold* or *differentiable manifold* or *smooth manifold* is a pair (M, A) where A is a maximal atlas of charts for M. The atlas is referred to as the *differentiable structure*.

Definition $f : M \to N$ is a *diffeomorphism* if it is one-to-one, onto and x is in N's atlas if and only if $x \circ f$ is in M's atlas.

Definition $f : M \to N$ is called *differentiable* if for every coordinate system (x, U) for M and (y, V) for N, the map $y \circ f \circ x^{-1} : \mathbf{R}^n \to \mathbf{R}^m$ is differentiable.

Bump Functions Define $h : \mathbf{R} \to \mathbf{R}$ by

$$h(x) = \begin{cases} e^{-1/x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$h(x)$ is C^∞, and $h^{(n)}(0) = 0$ for all n. Define $j : \mathbf{R} \to R$ by

$$j(x) = \begin{cases} h(x-1)h(x+1) & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases} = \begin{cases} e^{-(x-1)^{-2}} \cdot e^{-(x+1)^{-2}} & x \in (-1,1) \\ 0 & \text{otherwise} \end{cases}$$

We may shift and scale j to get $k : \mathbf{R} \to \mathbf{R}$ which is positive on $(0,1)$ and 0 elsewhere by $k(x) = j(2(x-1))$. Next define $l : \mathbf{R} \to \mathbf{R}$ by $l(x) = (\int_0^x k) / \left(\int_0^1 k\right)$. This is C^∞, increasing, 0 for $x \leq 0$, and 1 for $x \geq 1$. Finally define $g : \mathbf{R}^n \to \mathbf{R}$ to be the product of $j(x^1/\epsilon) \cdot j(x^n/\epsilon)$ which is positive on $(-\epsilon, \epsilon) \times \cdots \times (-\epsilon, \epsilon)$ and 0 elsewhere.

Lemma *Let $C \subset U \subset M$ with C compact and U open. Then there is a C^∞ function $f : M \to [0,1]$ such that $f = 1$ on C and U contains the support (the closure of the set $\{x : f(x) \neq 0\}$) of f.*

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Proposition *For any differentiable $f, g : M \to \mathbf{R}$, and any coordinate system (x, U) with $p \in U$, the operator $l = \partial/\partial x^i|p$ satisfies*

$$l(fg) = f(p)l(g) + l(f)g(p)$$

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Definition If $f : M^m \to N^n$ is C^∞ and (y, V) is a coordinate system around $f(p)$, the rank of the $n \times m$ matrix

$$\left(\frac{\partial(y^i \circ f)}{\partial x^j}(p)\right)$$

does not depend on the coordinate systems (x, U) or (y, V), and is called the *rank of f at p*. The point $p$is called a *critical point* of f if the rank of f at p is $< n$ (the dimension of the image N); if p is not a critical point, it is called a *regular point* of f. If p is a critical point, then $f(p)$ is a *critical value*. v is a *regular value* if $f^{-1}(v)$ consists only of regular points.

Lemma *If $f : \mathbf{R}^n \to \mathbf{R}^n$ is C^1 and A has measure 0, then $f(A)$ has measure 0.*

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Cor *If $f : M \to N$ is a C^1 function between two n-manifolds and $A \subset M$ has measure 0, then $f(A) \subset N$ has measure 0.*

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Sard's Theorem *If $f : M \to N$ is a C^1 map between n-manifolds, and m has at most countably many components, then the critical values of f form a set of measure 0 in N.*

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Stronger Sard's Theorem *The critical values of a C^k map $f : M^m \to N^n$ are a set of measure 0 if $k \geq 1 + \max(m-n, 0)$.*

Above is easy case, $m > n$ is trivial case. Look in Milnor, *Topology From the Differentiable Viewpoint* or Sternberg, *Lectures on Differential Geometry.* Page 55

Theorem *(1) If $n \leq m$ and $f : M^m \to N^n$ has rank n at p, then for any coordinate system (y, V) around $f(p)$, there is some coordinate system (x, U) around P with*

$$y \circ f \circ x^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^n)$$

(2) If $m \leq n$ and $f : M^m \to N^n$ has rank m at p, then for any coordinate system (x, U) around p, there is a coordinate system (y, V) around $f(p)$ with

$$y \circ f \circ x^{-1}(a^1, \ldots, a^m) = (a^1, \ldots, a^m, 0, \ldots, 0)$$

on some subneighborhood U' of U.

Page 59. Note: book should be corrected to include bit about U' above.

Definition A differentiable function $f : M^m \to N^n$ is called an *immersion* if the rank of f is m, the dimension of the domain, at all points of M. Immersions are locally one-to-one, though there exist differentiable globally one-to-one maps which are not immersions.

Definition A subset $M_1 \subset M$ with a differentiable structure (not necessarily compatible with the metric inheritted from M) is called an *Immersed submanifold* of M if the inclusion map $i : M_1 \to M$ is an immersion. Note (irrational slope line on a torus) that M_1 may even be a dense subset. Also note that if $f : P \to M$ is C^∞, it may be that f is not even continuous as a map into M_1. Page 64: Why this is the only thing that can go wrong.

Definition If $f : P \to M$ is a one-to-one immersion which is a homeomorphism onto its range is called an *imbedding* or *embedding*. An immersed submanifold is called a *(C^∞) submanifold* if the inclusion map is an imbedding. It is called a *closed submanifold* if it is also a closed subset of M.

Inverse Function Theorem *Suppose $W \subset \mathbf{R}^n$ is open, $h : W \to R^n$ is C^∞, $a \in W$ and suppose $(Dh)_a : \mathbf{R}^n \to \mathbf{R}^n$ is an isomorphism. Then there exists a neighborhood Z of a in W such that $h(Z)$ open in $\mathbf{R}^n$, $h|_Z \to h(Z)$ is a C^∞diffeomorphism (so $h^{-1}|_{h(Z)} \to Z$ is also C^∞).*

Theorem *If $f : M^m \to N$ has constant rank k on a neighborhood of $f^{-1}(y)$, then $f^{-1}(y)$ is a closed submanifold of M of dimension $m-k$ (or is empty). In particular, if y is a regular value of $f : M^m \to N^n$, then $f^{-1}(y)$ is an $(m-n)$-dimensional submanifold of M (or is empty).*

Proof of special case:

Inverse Image of a Regular Value *If $f : M^m \to N^n$ is C^∞ and $b \in N$ is a regular value of f, then $f^{-1}(b)$ is a C^∞ submanifold of dimension $k = m - n$ or is empty.*

Proof (Uses Inverse Function Theorem) Let $a \in f^{-1}(b)$. b is a regular value, so a is a regular point and f has rank n at a. Let $\phi : U \to \mathbf{R}^m$ be a chart for M with $\phi(a) = 0$, $\psi : V \to \mathbf{R}^n$ be a chart for N with $\psi(b) = 0$, so that $g \equiv \psi f \phi^{-1} : \phi(U) \to \mathbf{R}^n$ is C^∞ (wlog $f(U) \subset V$). It suffices to prove that there exists a neighborhood W of 0 in $\phi(U)$ such that $W \cap g^{-1}(0)$ is a C^∞ submanifold of the open set $W \subset \mathbf{R}^m$, dimension $k = m - n$.

f has rank n at $a \Longrightarrow (Dg)_0 : \mathbf{R}^m \to R^n$ has rank $n \Longrightarrow (Dg)_0$ is *ONTO.* So there exists a linear map $\gamma : \mathbf{R}^m \to \mathbf{R}^k$ such that $(Dg)_0 \oplus \gamma : \mathbf{R}^m \to \mathbf{R}^n \oplus \mathbf{R}^k$ is an isomorphism. Define $h : \phi(U) \to \mathbf{R}^n \times \mathbf{R}^k$ by $h(t) = (g(t), \lambda(t))$. Then $(Dh)_0 = (Dg)_0 \oplus \lambda$ is an isomorphism, so h maps some neighborhood W of 0 in $\phi(U)$ diffeomorphicallly onto open $h(W) \subset \mathbf{R}^n \times \mathbf{R}^k$ by the Inverse Function Theorem. Therefore $W \cap g^{-1}(0) = W \cap h^{-1}(0 \times \mathbf{R}^k)$ is a C^∞ submanifold of W of dimension $k = m - n$. $\surd$

Theorem *If $\mathcal{O}$ is an open cover of a manifold M, then there is an open cover $\mathcal{O}'$ of M which is locally finite and which refines $\mathcal{O}$. Moreover, we can choose all members of $\mathcal{O}'$ to be open sets diffeomorphic to $\mathbf{R}^m$.*

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Shrinking Lemma *Let $\mathcal{O}$ be a open locally finite cover of a manifold M. Then it is possible to choose, for each U in $\mathcal{O}$, an open set U' with $\overline{U}' \subset U$ in such a way that the collection of all U' is also an open cover of M.*

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Partitions of Unity *If $\mathcal{O}$ is any open cover of a manifold M, then there is a collection of C^∞ functions $\phi_i : M \to [0,1]$ such that:*

(1) the collection of sets $\{p : \phi_i(p) \neq 0\}$ is locally finite

(2) $\sum_i \phi_i(p) = 1$ for all $p \in M$

(3) for each i there is a $U \in \mathcal{O}$ such that $\operatorname{supp} \phi_i \subset U$.

Definition A collection $\{\phi_i : M \to [0,1]\}$ satisfying (1) and (2) is called a *Partition of unity*; if it satisfies (3), it is called *subordinate* to $\mathcal{O}$.

Page 68–70.

Theorem *If M^m is a compact C^∞ manifold, then there is an imbedding $f : M \to \mathbf{R}^N$ for some N.*

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The Tangent Bundle

Define $T\mathbf{R}^n$ as $\mathbf{R}^n \times \mathbf{R}^n$ with π projection onto the first coordinate. If $f : \mathbf{R}^n \to \mathbf{R}^m$ is a differentiable map and $p \in \mathbf{R}^n$, then there is a linear transformation $Df(p) : R^n \to R^m$ which may be used to produce a linear map from $\mathbf{R}^n_p \to \mathbf{R}^m_{f(p)}$ defined by

$$f_{*p} : v_p \mapsto [DF(p)(v)]_{f(p)}$$

The symbol f_* denotes the map $f_* : T\mathbf{R}^n \to T\mathbf{R}^m$ which is the union of all f_{*p}. The following diagram commutes ($\pi \circ f_* = f \circ \pi$):

$$\begin{array}{ccc} T\mathbf{R}^n & \xrightarrow{f_*} & TR^m \\ {\scriptstyle\pi}\downarrow & & \downarrow{\scriptstyle\pi} \\ \mathbf{R}^n & \xrightarrow{f} & \mathbf{R}^m \end{array}$$

Suppose $g : \mathbf{R}^m \to \mathbf{R}^k$ then $g_* \circ f_* = (g \circ f)_*$. Page 89.

Consider manifold M^m and imbedding $i : M \to \mathbf{R}^N$, and coordinate system (x, U) around $p \in M$. Then $i \circ x^{-1} : \mathbf{R}^m \to \mathbf{R}^N$ has rank m. Consequently, $(i \circ x^{-1})_*(\mathbf{R}^n_{x(p)})$ is an n-dimensional subspace of $\mathbf{R}^N_{i(p)}$ which does not depend upon the choice of coordinate system x (pages 91–92). Define $T(M, i)$ to be the disjoint union $\bigcup_{p \in M} (M, i)_p \subset i(M) \times \mathbf{R}^N \subset T\mathbf{R}^N$, and π to be the projection map $T(M, i) \to M$ by $\pi(v) = p$ if $v \in (M, i)_p$. Note $\pi^{-1}(p)$ has a vector space structure.

For any imbedding $i : M \to \mathbf{R}^N$, the structure of $T(M, i)$ is always simple locally. If (x, U) is a coordinate system on M, then $\pi^{-1}(U)$ can be mapped homeomorphically onto

$U\times\mathbf{R}^m$. $T(M,i)$ is said to be "locally trivial," and is an example of an *m-dimensional vector bundle* or *m-plane bundle.* $\pi : E\to B$ is a m-dimensional vector bundle if π is continuous and onto, and every $p\in B$ has a neighborhood $U\ni p$ such that $\pi^{-1}(U)$ is homeomorphic to $U\times\mathbf{R}^n$ via a vector space isomorphism of fibers.

Two vector bundles $\pi_1 : E_1\to B$ and $\pi_2 : E_2\to B$ are *equivalent* if there is a homeomorphism taking $E_1\to E_2$ which takes $\pi_1^{-1}(p)$ isomorphically to $\pi_2^{-1}(p)$. A *bundle map* from $\pi_1 : E_1\to B_1$ to $\pi_2 : E_2\to B_2$ is a pair of continuous maps $(\tilde{f},f)$, $\tilde{f} : E_1\to E_2$, $f : B_1\to B_2$ such that

$$\begin{array}{ccc} E_1 & \xrightarrow{\tilde{f}} & E_2 \\ {\scriptstyle\pi_1}\downarrow & & \downarrow{\scriptstyle\pi_2} \\ B_1 & \xrightarrow{f} & B_2 \end{array}$$

commutes, and $\tilde{f} : \pi_1^{-1}(p)\to\pi_2^{-1}(f(p))$ is a linear map. If $f : \mathbf{R}^k\to\mathbf{R}^l$ is differentiable, the pair (f_*,f) is a bundle map from $T\mathbf{R}^k$ to $T\mathbf{R}^l$.

Alternatively, we could define a tangent vector at p to be a linear operator l which operates on all C^∞ functions f and which is a *derivation at p*:

$$l(fg)(p) = f(p)l(g) + g(p)l(f)$$

Recall the operators $l = \partial/\partial x^i|p$ have this property. $l(f) = l(g)$ if $f = g$ in a neighborhood of p (Page 106). The set of all such operators is a vector space.

Lemma *Let f be a C^∞ function in a convex open neighborhood U of 0 in $\mathbf{R}^n$, with $f(0) = 0$. Then there are C^∞ functions $g_i : U\to\mathbf{R}$ with $f(x^1,\ldots,x^n) = \sum_{i=1}^n x^i g_i(x^1,\ldots,x^n)$ for $x\in U$ and $g_i(0) = D_i f(0)$ (the second condition actually follows from the first).*

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Theorem *The set of all linear derivations at $p\in M^m$ is an m-dimensional vector space. In fact, if (x,U) is a coordinate system around p, then*

$$\left.\frac{\partial}{\partial x^1}\right|_p,\cdots\left.\frac{\partial}{\partial x^m}\right|_p$$

span this vectorspace, and any derivation l can be written

$$l = \sum_{i=1}^m l(x^i)\cdot\left.\frac{\partial}{\partial x^i}\right|_p$$

(so l is determined by the numbers $l(x^i)$).

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Note: A tangent vector is often most easily described by telling what derivation it corresponds to, and the map f_* is often most easily anlysed from the relation $(f_*v)(g) = v(g\circ f)$.

On $\mathbf{R}^1$ write $d/dt|t_0$ for $\partial/\partial t|t_0$ as the basis for $\mathbf{R}_{t_0}$. Let $c : \mathbf{R}\to M$ be a differentiable curve, then

$$c_*\left(\left.\frac{d}{dt}\right|_{t_0}\right)\in M_{c(t_0)}$$

is called the *tangent vector to c at t_0*, written

$$\left.\frac{dc}{dt}\right|_{t_0}.$$

Observe TM is locally $U \times \mathbf{R}^m$, and is therefore a manifold. It has a natural C^∞ structure: given $x : U \to \mathbf{R}^m$ a chart on M, then $v \in (TM)|U = \pi^{-1}(U)$ may be represented (uniquely)

$$v = \sum_{i=1}^{m} a^i \cdot \left.\frac{\partial}{\partial x^i}\right|_p$$

for $p = \pi(v) \in U$. If we denote a^i by $\dot{x}^i(v)$ then the map $v \mapsto (x(\pi(v)), \dot{x}) \in \mathbf{R}^{2m}$ is a homeomorphism from $(TM)|U$ to $U \times \mathbf{R}^m$. If y is another coordinate system, then $y_* \circ (x_*)^{-1}$ is C^∞ under this C^∞ structure.

A *section* of a bundle $\pi : E \to B$ is a continuous function $s : B \to E$ such that $\pi \circ s$ is the identity on B. A C^∞ section of TM is called a *vector field* on M. For any coordinate system (x, U) we have

$$X(p) = \sum_{i=1}^{m} a^i(p) \cdot \left.\frac{\partial}{\partial x^i}\right|_p$$

for all $p \in U$, and the a^i are C^∞ if and only if $X : U \to TM$ is C^∞. If $f : M \to \mathbf{R}$ define the vector field fX by $(fX)(p) = f(p)X(p)$. Let $\partial/\partial x^i$ denote the vector field

$$p \mapsto \left.\frac{\partial}{\partial x^i}\right|_p .$$

Define the function $\bar{X}(f) : M \to \mathbf{R}$ by $\bar{X}(f)(p) = X_p(f)$. If locally,

$$X(p) = \sum_{i=1}^{m} a^i(p) \cdot \left.\frac{\partial}{\partial x^i}\right|_p$$

then

$$\bar{X}(f) = \sum_{i=1}^{m} a^i t \frac{\partial f}{\partial x^i}$$

Let $\mathcal{F}$ denote the set of all C^∞ functions on M. A C^∞ vector field X gives rise to a function $\bar{X} : \mathcal{F} \to \mathcal{F}$ satisfying $\bar{X}(f + g) = \bar{X}(f) + \bar{X}(g)$ and $\bar{X}(fg) = f\bar{X}(g) + g\bar{X}(f)$. Thus $\bar{X}$ is a "derivation" of the ring $\mathcal{F}$.

Orientation

Two ordered bases $(v_1, \ldots, v_n)$ and $(v_1', \ldots, v_n')$ determine an isomorphism $f : V \to V$ with $f(v_i) = v_i'$. We call $(v_1, \ldots, v_n)$ and $(v_1', \ldots, v_n')$ *equally oriented* if $\det f > 0$ and *oppositely oriented* if $\det f < 0$. This divides orderd basis into two equivalence classes called *orientations*. If (V, μ) and (W, ν) are two n-dimensional oriented vector spaces, an isomorphism $f : V \to W$ is called *orientation preserving (with respect to μ and ν)* if $[f(v_1), \ldots, f(v_n)] = \nu$ whenever $[v_1, \ldots, v_n] = \mu$ (well-defined independent of choice of v_i). Endow the trivial bundle $X \times \mathbf{R}^n$ with the "standard orientation" $[(x, e_1), \ldots, (x, e_n)]$ on each fiber $\{x\} \times \mathbf{R}^n$. If $\pi : E \to B$ is a non-trivial n-plane bundle, an *orientation* μ of E is defined to be a collection of orientations μ_p for $\pi^{-1}(p)$ which satisfy, for any open connected $U \subset B$: If $t : \pi^{-1}(U) \to U \times \mathbf{R}^n$ is an equivalence, and the fibers of $U \times \mathbf{R}^n$ are given the standard orientation, then t is either orientation preserving or orientation reversing on all fibers. More details Pages 115–117.

Definition A bundle is called *orientable* if it has an orientation; an *oriented bundle* is just a pair (ξ, μ) where μ is an orientation for ξ. Applying this to TM of a C^∞ manifold M, we call M *orientable* if the bundle TM is orientable. Note (page 119): for any orientable bundle, the restriction to any subset is also orientable.

Definition (Alternative) M is orientable if there is a subset $\mathcal{A}'$ of the atlas $\mathcal{A}$ for M such that the domains of $(x, U) \in \mathcal{A}'$ cover M, for all $(x, U), (y, V) \in \mathcal{A}'$, $\det(\partial y^i/\partial x^j) > 0$ on $U \cap V$. Pages 120–121.

Tensors

Definition Given a vector space V, let the *dual space* V^* is the vector space of all linear functions $\lambda : V \to \mathbf{R}$. If $f : V \to W$ is a linear transformation $f^* : W^* \to V^*$ is defined by

$$(f^*\lambda)(v) = \lambda(fv).$$

Note that $(f \circ g)^* = g^* \circ f^*$. If v_i is a basis for V, then v_i^* defined by $v_i^*(v_j) = \delta_{ij}$ is a basis for V^*.

Definition Let $\xi = \pi : E \to B$ be a vector bundle. Define $E' = \bigcup_{p \in B} [\pi^{-1}(p)]^*$, and $\pi' : E' \to B$ to take $[\pi^{-1}(p)]^*$ to p. By requiring local trivializations of ξ to be local trivializations of $\xi^* = \pi' : E' \to B$ make ξ' into a vector bundle, called the *dual bundle* of ξ.

If s is a section of ξ and σ is a section of ξ^*, then we can define a function $\sigma(s)$ from $B \to \mathbf{R}$ by $p \mapsto \sigma(p)(s(p))$ since $s(p) \in \pi^{-1}(p)$ and $\sigma(p) \in \pi'^{-1}(p) = \pi^{-1}(p)^*$.

Definition Taking the dual bundle of the tangent bundle TM gives the *cotangent bundle* T^*M.

If ω is a C^∞ section of T^*M and X is a C^∞ vector field, then $\omega(X)$ is the C^∞ function $p \mapsto \omega(p)(X(p))$. If $f : M \to \mathbf{R}$ is a C^∞ function, then a C^∞ section df of T^*M can be defined by $df(p)(X) = X(f)$ for $X \in M_p$. This section is called the *differential of f*. See page 151.

Theorem *If (x, U) is a coordinate system and f is a C^∞ function, then on U we have*

$$df = \sum_{i=1}^{n} \frac{\partial f}{\partial x^i} dx^i .$$

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Vector Fields and Differential Equations

Let X be a vector field defined in a neighborhood of $p \in M$. Want curve $\rho : (-\epsilon, \epsilon) \to M$ through p whose tangent agrees with X, ie. $\rho(0) = p$ and

$$\rho_* \left(\left. \frac{d}{dt} \right|_t \right) = \left. \frac{d}{dt} \right|_t = X(\rho(t)).$$

Since this is a local question, want to introduce a coordinate system (x, U) around p. If α is a diffeomorphism then define

$$(\alpha_* X)_q = \alpha_*(X_{\alpha^{-1}}) = \alpha_{*_{\alpha^{-1}(q)}}(X_{\alpha^{-1}})$$

and $\alpha_* X$ is C^∞ on $\alpha(M)$. In particular, this defines $x_* X$ on $x(U) \subset \mathbf{R}^n$. Define component functions $f^1(q), \ldots, f^n(q)$ of $(x_* X)_q$, so $f(q) = (x_* X)_q$. Now consider $c = x \circ \rho$, so

$$\left.\frac{dc}{dt}\right| = x_* \rho_* \left(\left.\frac{d}{dt}\right|_t \right) = x_*(X(\rho(t))) = (x_* X)_{x(\rho(t))} = (x_* X)_{c(t)}$$

or more simply $c'(t) = f(c(t))$. This is equivalent to a system of n differential equations

$$c^{i\prime}(t) = f^i(c^1(t), \ldots, c^n(t)) \text{ with } c^i(0) = x^i(p).$$

Definition A curve $\rho : (-\epsilon, \epsilon) \to M$ such that $\rho(0) = p$ and $d\rho/dt = X(\rho(t))$, is called an *integral curve* for X with *initial condition* $\rho(0) = p$.

We have shown the existence of integral curves on M is equivalent to the problem in $\mathbf{R}^n$, where we will refer to open $U \subset \mathbf{R}^n$, $f : U \to \mathbf{R}^n$, and a curve $c : (-\epsilon, \epsilon) \to \mathbf{R}^n$ with $c(0) = x$ and $c'(t) = f(c(t))$ as an *integral curve* for f.

Example Pages 188–190

Definition In order to get unique solutions to the system of differential equations, consider the *Lipshitz condition* for f on U, that there is some K such that $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in U$. Note that Lipschitz implies continuous but not necessarily differentiable, and that C^1 implies Lipschitz locally.

Contraction Lemma *Let (M, ρ) be a non-empty complete metric space, and let $f : M \to M$ be a "contraction," ie. there is some $C < 1$ such that $\rho(f(x), f(y)) \le C\rho(x, y)$ for all $x, y \in M$. Then there is a unique fixed point of f, ie. x such that $f(x) = x$.*

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Idea: Apply this theorem to the metric space of continuous functions from X to M (where either X is compact or M is bounded) with the sup norm. Note that M complete implies the metric space is complete (uniform limit of continuous functions is continuous). Especially consider M a compact subset of $\mathbf{R}^n$.

Theorem *Let $f : U \to \mathbf{R}^n$ be any function with U open. Fix $x_0 \in U$ and $a > 0$ such that $\bar{B}_{2a}(x_0) \subset U$. Suppose that $|f| \le L$ on $\bar{B}_{2a}(x_0)$ and $|f(x) - f(y)| \le K|x - y|$ for all $x, y \in \bar{B}_{2a}(x_0)$. Given $0 < b \le \min(a/L, 1/K)$, then for each $x \in \bar{B}_{2a}(x_0)$ there is a unique $\alpha_x : (-b, b) \to U$ such that $\alpha_x'(t) = f(\alpha_x(t))$ and $\alpha_x(0) = x$.*

Proof Page 193–197. Want to define $\alpha_x(t) = x + \int_0^t f(\alpha(u))\, du$ (defined componentwise) and use the estimate that

$$\left| \int_a^b f(u)\, du \right| \le L(b - a).$$

Proof shows the estimate true for constant functions, hence for step functions, and therefore continuous functions (which are uniform limits of step-functions).

To accomplish this define $S\alpha(t) = x + \int_0^t f(\alpha(u))\, du$ for $t \in (-b, b)$, then use the estimate to show $S\alpha(t) \in \bar{B}_{2a}(x_0)$. In fact S is a contraction mapping on the space of possible αs, so it has a unique fixed point, which after some finicky details is shown to be a solution. $\surd$

In fact this solution is unique, see pages 197-198.

The map $\alpha(t,x) = \alpha_x(t) : (-b,b) \times B_a(x_0) \to U$ satisfys $\alpha(0,x) = x$ and

$$\frac{d}{dt}\alpha(t,x) = f(\alpha(t,x))$$

and is therefore called a *local flow* for f in $(-b,b) \times B_a(x_0)$.

If f is locally Lipschitz, then α given by the above theorem is continuous. See pages 198–200. It is a hard theorem that if f is C^k then the flow α is also C^k.

Note that we may fix t to get the map $\phi_t(x) = \alpha(t,x)$. If α is C^∞, then ϕ will also be C^∞.

Theorem *Let X be a C^∞ vector field on M, and let $p \in M$. Then there is an open set V containing p and an $\epsilon > 0$, such that there is a unique collection of diffeomorphisms $\phi_t : V \to \phi_t(V) \subset M$ for $|t| < \epsilon$ with the following properties:*

(1) $\phi : (-\epsilon,\epsilon) \times V \to M$ defined by $\phi(t,x) = \phi_t(x)$ is C^∞.

(2) If $|s|, |t|, |s+t| < \epsilon$, and $q, \phi_t(q) \in V$, then $\phi_{s+t}(q) = \phi_s \circ \phi_t(q)$.

(3) If $q \in V$, then X_q is the tangent vector at $t = 0$ of the curve $t \mapsto \phi_t(q)$.

Pages 201–203.

Definition The *support* of a vector field X is just the closure of $\{p \in M : X_p \neq 0\}$.

Theorem *If X has compact support (or if M is compact) then there are diffeomorphisms $\phi_t : M \to M$ for all $t \in \mathbf{R}$ with properties (1), (2), and (3) above.*

Proof Compactness and checking overlaps are the same by uniqueness argument. Pages 203–204. $\surd$

Remark A C^∞ flow ϕ on M determines a C^∞ vector field X on M, called the *velocity vector field* of ϕ, or the *vector field generated by* ϕ. For $x \in M$ set $X(x) = (\alpha \mapsto \frac{d}{dt}(\alpha(\phi(x,t)))_{t=0})$ the velocity vector of curve $t \mapsto \phi_t(x)$ at $t = 0$. The flow being $C^\infty \implies$ the vector field is C^∞.

Note Know M closed C^∞ manifold and X a C^∞ vector field $\implies$ there exists a C^∞ flow. Closed is important since imagine a vector field determined by rotation of the sphere around the z-axis, if you remove a point from the sphere, the flow can not be defined for both all points and all time.

Remark Consider studying the Isotopy Extension Theorem to guard against Casson asking things not on the syllabus. See notes 10-23 (2) through 10-28 (1).

Lie Brackets

Definition Let $X : M \to TM$ be a function (not necessarily C^∞) such that $\forall x \in M, X(x) \in T_xM = \{\text{derivations on } M \text{ at } x\}$. For each $\alpha \in C^\infty(M) = C^\infty(M,\mathbf{R})$ define $X_\alpha : M \to \mathbf{R}$ by $(X\alpha)(x) = (X(x))(\alpha)$.

Lemma *X is a C^∞ vector field on M iff $\forall a \in C^\infty(M)$, $X\alpha : M \to \mathbf{R}$ is C^∞.*

Corollary *C^∞ vector fields on M are just linear maps $X : C^\infty(M) \to C^\infty(M)$ such that $\forall \alpha, \beta \in C^\infty(M)$, $X(\alpha\beta) = \alpha X(\beta) + \beta X(\alpha)$. This correspondence is given by $X \mapsto (\alpha \mapsto X\alpha)$.*

Notes 10-28 (2).

Suppose X, Y are C^∞ vector fields on M, $X, Y : C^\infty(M) \to C^\infty(M)$. Is the composition $X \circ Y : C^\infty(M) \to C^\infty(M)$ a derivation/vector field? Check the product rule. In general it won't, but the *Lie Bracket* $[X,Y] = XY - YX$ does satsify the product rule, so $[X,Y]$ is a C^∞ vector field assuming X, and Y are.

Jacobi Identity *For any C^∞ vector fields X, Y, Z*

$$[X,[Y,Z]] + [Y,[Z,X]] + [Z,[X,Y]] = 0$$

Homework 3.

Example Let $M = \mathbf{R}^n$, $\partial_i = \frac{\partial}{\partial x^i}$, then $[\partial_i, \partial_j] = 0$ (by the symmetry of 2nd partial derivatives).

Example Suppose X, Y are C^∞ vector fields on M and $\alpha, \beta : M \to \mathbf{R}$ are C^∞. Then $\alpha X, \beta Y$ are C^∞ vector fields on M $(\alpha X)(x) = \alpha(x)X(x) \in T_x(M)$.

$$[\alpha X, \beta Y] = \alpha X(\beta)Y + \beta\alpha(XY) - \beta Y(\alpha)X - \alpha\beta YX = \alpha X(\beta)Y - \beta Y(\alpha)X + \alpha\beta[X,Y]$$

Example Using the above two examples, can calculate Lie bracket of any two C^∞ vector fields on $\mathbf{R}^n$.

Definition Let $f : M \to N$ be a C^∞ map of C^∞ manifolds. Let X be a C^∞ vector field on M, Y a C^∞ vector field on N. Then X, Y are *f-related* if for all $x \in M$, $(Df)_x(X(x)) = Y(f(x))$.

Lemma *Vector fields X on M, Y on N are f-related iff $\forall C^\infty$ $\alpha : N \to \mathbf{R}$, $(Y\alpha) \circ f = X(\alpha \circ f) : M \to \mathbf{R}$.*

Notes 10-30 (2).

Proposition *Let $f : M \to N$ be a C^∞ function such that f is an immersion. If Y is a C^∞ vector field on N with $Y_{f(p)} \in f_{p*}(M_p)$, then there is a unique C^∞ vector field X on M which is f-related to Y.*

Spivak pages 258–259.

Lemma *Suppose X_i, Y_i are f-related for $i = 1, 2$. Then $[X_1, X_2]$ and $[Y_1, Y_2]$ are f-related.*

Notes 10-30 (2).

Recall If X C^∞ vector field on M, $a \in M$ then $\exists \epsilon > 0$ and unique integral curve $u : (-\epsilon, \epsilon) \to M$ such that $u(0) = a$ and $u'(t) = X(u(t))$ for $t \in (-\epsilon, \epsilon)$. Note that in this case $\frac{\partial}{\partial t}$ and X are u-related.

Lemma *Suppose X and Y are C^∞ vector fields on M, and $a \in M$. Then $\exists \epsilon > 0$ and (unique) map $u : (-\epsilon, \epsilon) \times (-\epsilon, \epsilon) \to M$ such that $u(0,0) = a$ and $\frac{\partial}{\partial s}, \frac{\partial}{\partial t}$ are u-related to X, Y respectively* provided $[X,Y] = 0$ *in some neighborhood of a.*

Notes 10-30 (2) through 11-1 (2).

Lemma *Let $X_1, \ldots, X_k$ be C^∞ vector fields on M, and $a \in M$. Then there exists $\epsilon > 0$ and C^∞ map $u : (-\epsilon, \epsilon)^k \to M$ such that $u(0,0) = a$ and $\frac{\partial}{\partial t_i}$ is u-related to X_i for $1 \le i \le k$,* provided $[X_i, X_j] = 0 \forall i, j$.

Proof Induction starting at $k = 2$. $\checkmark$

Remark If X, Y are velocity vector fields for flows ϕ, ψ, M closed, then $[X,Y] = 0$ *iff* ϕ_s, ψ_t commute $\forall s, t$.

Spivack page 217.

Distributions and Frobenius Theorem

Definition A *k-dimensional distribution* Δ on a C^∞ manifold M assigns to each point $x \in M$ a k-dimensional subspace $\Delta_x \subset T_xM$. Δ is a *C^∞ distribution* if, for each point $a \in M$, there exists a neighborhood U of a and C^∞ vector fields $X_1, \dots, X_k$ on U such that, for each $x \in U$ $X_1(x), \dots, X_k(x)$ are a basis for $\Delta_x \subset T_xM$.

Definition An *integral manifold* for k-dim distribution Δ on M is an immersed submanifold (image of a one-to-one C^∞immersion) $N^k \subset M$ such that $\forall a \in N$, $T_aN = \Delta_a \subset T_aM$.

Definition A vector field X (*is tangent to*) *belongs to distributuion* Δ on M iff $\forall x \in M$, $X(x) \in \Delta_x \subseteq T_xM$.

Frobenius Theorem *Let Δ be a C^∞ distribution on M. Then Δ has an integral manifold through each point of M iff X, Y C^∞ vector fields belonging to $\Delta \Longrightarrow [X,Y]$ belongs to Δ.*

See 11-4 (2) to 11-6 (1).

Definition Δ is *integrable* if either of these equivalent conditions is satisfied.

Example

$$\left[\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x_1\frac{\partial}{\partial x_2}\right] = \frac{\partial}{\partial x_3} \notin \langle\frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2} + x\frac{\partial}{\partial x_3}\rangle$$

Note 1-dim distributions always satisfy $[,]$ condition so are always integrable.

Addendum If Δ is an integrable C^∞ k-dim distribution on M, then for each $a \in M$, there exists a chart $\phi : U \to \mathbf{R}^k \times \mathbf{R}^{m-k}$ such that $\phi(U) = \mathbf{R}^k \times \mathbf{R}^{m-k}$ and, for each $y \in \mathbf{R}^{m-k}$, $\phi^{-1}(\mathbf{R}^k \times \{y\})$ is an intetral submanifold for Δ.

This says that integral submanifolds through a "depend on a in a C^∞ fashion."

See 11-6 (2).

Definition A k-dimesnional C^∞ *foliation* $\mathcal{F}$ on M^m consists of a C^∞ k-manifold N, a bijective immersion $i : N^k \to M^m$ such that, for each point $b \in N$, there exists a chart $\phi : U \to \mathbf{R}^k \times \mathbf{R}^{m-k}$ for M such that $\phi(i(b)) = 0$, $\phi(U) = \mathbf{R}^k \times \mathbf{R}^{m-k}$, and for each $y \in \mathbf{R}^{m-k}$, $i^{-1}\phi^{-1}(\mathbf{R}^k \times \{y\})$ is a component of $i^{-1}(U)$ (with respect to the topology on $N \cap U$ induced from N).

Definition The components of N are called *leaves* or *maximal integral manifolds* of $\mathcal{F}$.

Theorem *If Δ is a k-dim integrable C^∞ distribution on M^m, then Δ determines a k-dim foliation $\mathcal{F}$ on M: leaves of $\mathcal{F}$ are maximal integral manifolds of Δ.*

Set $N = M$ as a point set, let $i = 1 : N \to M$. Let $\mathcal{A}$ be the atlas of charts for M satisfying the conditions in the addendum to Frobenius' Theorem. For each chart in $\mathcal{A}$, say $\phi : U \to \mathbf{R}^k \times \mathbf{R}^{m-k}$ and each $y \in \mathbf{R}^{m-k}$, define $V = \phi^{-1}(\mathbf{R}^k \times \{y\})$ $V \subset N$ and $\phi = p_1 \circ \phi|_V \to \mathbf{R}^k$. Let $\mathcal{B}$ be the set of maps $\psi : V \to \mathbf{R}^k$ corresponding to all pairs $\phi \in \mathcal{A}$, $y \in \mathbf{R}^{m-k}$.

Lemma *If $\psi_i : V_i \to \mathbf{R}^k$ is in $\mathcal{B}$ ($i = 1, 2$) then $\psi_1(V_1 \cap V_2)$ is open in $\mathbf{R}^k$ and $\psi_2\psi_1^{-1} : \psi_1(V_1 \cap V_2) \to \mathbf{R}^k$ is C^∞.*

See 11-8 (1–2) in Notes.

Corollary *N is a k-manifold where $W \subset N$ is open iff $\forall \psi : V \to \mathbf{R}^k$ in $\mathcal{B}$, $\psi(V \cap W)$ open in $\mathbf{R}^k$ and a C^∞ manifold, with $\mathcal{B}$ as defining atlas.*

Transversality

Definition Let P^p be a C^∞ submanifold of N^n, and let M^m be any C^∞ manifold. A C^∞ map $f : M \to N$ is *transverse to* P (written $f \pitchfork P$) if, for all $x \in M \cap f^{-1}(P)$, $T_{f(x)}N$ is spanned by $(Df)_x(T_xM)$ and $T_{f(x)}P$. Ie. $T_{f(x)}N = (Df)_x(T_xM) + T_{f(x)}P$.

Theorem *If $f : M^m \to N^n$ is transverse to $P^p \subseteq N^n$, a C^∞ submanifold, then $f^{-1}(P)$ is a C^∞ submanifold of M with dimension* $\dim M - \dim N + \dim P = m - n + p$.

Proof From Homework 4. We will show that $f^{-1}(P) \cap U$, U open in M, is a submanifold for all U in an open cover of $f^{-1}(P)$. Fix $v \in P \cap f(M)$. Since $y \in P$, there is some submanifold chart neighborhood, V, of y. Now let $U = f^{-1}(V) \subseteq M$, open since f is continuous. Notice that by varying y, we get a collection of Us in M which cover $f^{-1}(p)$. Since V is a submanifold chart neighborhood, there is a chart $\phi : V \to \mathbf{R}^n$ satisfying $\phi | V \cap P \to \mathbf{R}^p \times 0^{n-p}$. Define $\pi_2 : \mathbf{R}^n \to \mathbf{R}^{n-p}$ by $\pi_2(x_1, \ldots, x_p, x_{p+1}, \ldots, x_n) = (x_{p+1}, \ldots, x_n)$ - ie. projection onto the last $n - p$ components. Claim: $0 \in \mathbf{R}^{n-p}$ is a regular value of $g \equiv \pi_2 \circ \phi \circ f : U \to \mathbf{R}^{n-p}$, ie. g has rank $n - p$ at all $x \in g^{-1}(0) = U \cap f^{-1}(p)$. Fix $x \in g^{-1}(0)$. Since $g(x) = 0$, $\phi(f(x)) \in \mathbf{R}^p \times 0^{n-p}$, and $f(x) \in V \cap P$. By the transversality of f, $T_{f(x)}N (\cong \mathbf{R}^n) = (Df)_x(T_xM) + T_{f(x)}P$. Now take ϕ_* of both sides, a linear isomorphism since ϕ is a diffeomorphism,

$$\phi_*(T_{f(x)}N) = T_{(\phi f)(x)}\mathbf{R}^n = \phi_*((Df)_x(T_xM) + T_{f(x)}P) = \phi_* f_*(T_xM) + \phi_* T_{f(x)}P$$

Finally taking π_{2*}

$$\begin{aligned} T_{(\pi_2 \phi f)(x)}\mathbf{R}^{n-p} &= \pi_{2*} T_{(\phi f)(x)}\mathbf{R}^{n-p} \\ &= \pi_{2*}(\phi_* f_*(T_xM) + \phi_* T_{f(x)}P) \\ &= \pi_{2*}\phi_* f_*(T_xM) + \pi_{2*}\phi_* T_{f(x)}P \end{aligned}$$

Notice $\pi_{2*}\phi_* T_{f(x)}P = 0$ for our choice of $x \in g^{-1}(0)$ since ϕ take P into $\mathbf{R}^p \times 0^{n-p}$ which projects to 0 under π_2. This and the chain rule gives us:

$$T_{(\pi_2 \phi f)(x)}\mathbf{R}^{n-p} = \pi_{2*}\phi_* f_*(T_xM) = (\pi_2 \circ \phi \circ f)_*(T_xM) = g_*(T_xM)$$

So g_* (and therefore g) has rank $n - p$ at x. By the theorem on regular value, $g^{-1}(0) = U \cap f^{-1}(P)$ is a (closed) submanifold of dimension $m - (n - p) = m + p - n$. Now taking the union over the Us (which cover $f^{-1}(p)$) gives us that $f^{-1}(P)$ is in fact a submanifold of dimension $m + p - n$. $\surd$

Definition Let M, N be C^∞ manifolds, let $P \subset N$ be a C^∞ submanifold. Let $A \subset M$ be any subset. Then C^∞map $f : M \to N$ is *transverse to P along A* (written $f \pitchfork_A P$) iff for all $x \in A \cap f^{-1}(P)$, $T_{f(x)}N = (Df)_x(T_xM) + T_{f(x)}P$.

Equip $C^\infty(M, N)$ with a topology such that (1) If M, N, T are C^∞ manifolds, $f : M \times T \to N$ C^∞ map, then $t \mapsto (x \mapsto f_t(x) = f(x, t))$ defines a continuous map $T \to C^\infty(M, N)$ and (2) if $P \subset N$ closed C^∞ submanifold, $\subset M$ compact set, then $\{f \in C^\infty(M, N) : f \pitchfork_A P\}$ is open in $C^\infty(M, N)$.

Transversality Theorem *Let M, N be C^∞ manifolds, $P \subset N$ be a closed C^∞ submanifold and let $A \subset M$ be compact. Then $\{f \in C^\infty(M, N) : f \pitchfork_A P\}$ is dense and open in $C^\infty(M, N)$ with respect to any topology on $C^\infty(M, N)$ satisfying (1) and (2) above (such as the C^1 topology below).*

Corrollary *If in addition M is compact, then $\{f \in C(M,N) : f \pitchfork P\}$ is dense open in $C^\infty(M,N)$.*

Lemma *Let $f : M \to N$ be a C^∞ map, let $P \subset N$ be a closed C^∞ submanifold, let $a \in M$, and let U be an open neighborhood of a in M. Then there exists a neighborhood A of a in U and $r > 0$ and C^∞ map $F : M \times B_r^k \to N$ (where $k = n - p$ is the dimension of the ball) such that:*

(1) $F_0 = f : M \to N$, where $F_t : M \to N$ defined by $F_t(x) = F(x,t)$ for $x \in M, t \in B_r^k$.

(2) $F_t(x) = f(x)$ for all t if $x \notin U$.

(3) For all $\delta > 0$, there exists $t \in B_r^k$ with $\|t\| < \delta$ such that $F_t : M \to N$ is transverse to f along A.

Proof (Lemma) If $f(a) \notin P$, P^c open, there exists a neighborhood A of a in U such that $f(A) \cap P = \emptyset$. Set $F_t(x) = F(x,t) = f(x) \forall t$, then (1)–(3) satisfied.

Suppose $f(a) \in P$. Let $\psi : V \to \mathbf{R}^n$ be a submanifold chart with $\psi(f(a)) = 0$, and $V \cap P = \psi^{-1}(\mathbf{R}^p \times 0)$ where $\mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^k$ $k = n - p$. Wlog $f(U) \subseteq V$ [cut down U if necessary]. Let A be a compact neighborhood of a in U. There exists $\phi : M \to [0,1]$ C^∞ such that $\operatorname{supp}(\phi) \subset U$ and $\phi(x) = 1$ for all $x \in A$, wlog $\operatorname{supp}(\phi)$ is compact. $f(\operatorname{supp}\phi) \subseteq f(U) \subseteq V$, so $\psi f(\operatorname{supp}\phi) \subseteq \psi(V)$ open in $\mathbf{R}^n$. There exists $r > 0$ such that a r-neighborhood of $\psi f(\operatorname{supp}\phi) \subseteq \psi(V)$. Define $F : M \to B_r^k \to N$ by

$$F(x,t) = \begin{cases} \psi^{-1}(\psi f(x) - t\phi(x)) & \forall x \in \operatorname{supp}\phi \\ f(x) & \text{for } x \notin \operatorname{supp}\phi \end{cases}$$

F is C^∞, and clearly satisfies (1) and (2).

Recall $\psi(V) \subset \mathbf{R}^n = \mathbf{R}^p \times \mathbf{R}^k$, $V \cap P = \psi^{-1}(\mathbf{R}^p \times 0)$. Let $p_2 : \mathbf{R}^n \to \mathbf{R}^k$ be projection $p_2(x_1, \ldots, x_n) = (x_{p+1}, \ldots, x_n) \in \mathbf{R}^k$. If $x \in A, t \in B_r^k$, then $\psi F_t(x) = \psi f(x) - t\phi(x) = \psi f(x) - t$ so $A \cap F_t^{-1}(P) = \{x \in A : \psi f(x) - t \in \mathbf{R}^p\} = \{x \in A : p_2 \psi f(x) = t\}$ and $F_t \pitchfork_A P \iff \psi F_t \pitchfork_A \mathbf{R}^p \iff \psi f \pitchfork_A p_2^{-1}(t) \iff p_2 \psi f \pitchfork_A t$. In particular, this holds if t is a regular value of $p_2\psi f : U \to \mathbf{R}^k$. By Sard's theorem, there exists arbitrarily small $t \in \mathbf{R}^k$ which are regular values of $p_2 \psi f : U \to \mathbf{R}^k$. $\surd$

Proof (Theorem) By condition (1) and lemma, for each $a \in M$ there exists a precompact open neighborhood W of a in M such that $\{f \in C^\infty(M,N) : f \pitchfork_{\bar{W}} P\}$ is dense in $C^\infty(M,N)$ (condition 1 says open sets in $C^\infty(M,N)$ have an inverse open image in $T = B_r^k$, Lemma says no matter how small we make r, it will still contain some point whose image is transverse on $\bar{W}$ to P). A compact, so there exists finitely many such Ws covering A, say $A = W_1 \cup \cdots \cup W_k$. For each r, $S_r = \{f \in C^\infty(M,N) : f \pitchfork_{\bar{W}_1 \cup \cdots \cup \bar{W}_r} P\}$ is *open* in $C^\infty(M,N)$ [by (2) and compactness of $\bar{W}_1 \cup \cdots \cup \bar{W}_r$]. Induction: S_r is also dense in $C^\infty(M,N)$. [Suppose S_{r-1} is dense and open: let $\emptyset \neq U \subset C^\infty(M,N)$ be open; must show $S_r \cap U \neq \emptyset$. Induct: $S_{r-1} \cap U$ non-empty open: by lemma, $T_r = \{f \in C^\infty(M,N) : f \pitchfork_{\bar{W}_r} P\}$ dense, $S_{r-1} \cap U \cap T_r \neq \emptyset$ but $S_r = S_{r-1} \cap T_r$ so $S_r \cap U \neq \emptyset$]. In particular $\{f \in C^\infty(M,N) : f \pitchfork_A P\} \supseteq S_k$ is dense in $C^\infty(M,N)$. (Actually, have to be careful to not destroy transversality on previous open sets when adding W_r, so we find t so that it is ok in a ball around that t. Now we restrict our future rs (or ts) to be small enough that it does not mess up our earlier W_r.) $\surd$

Definition For any compact $K \subset M$, open $U \subset N$, set $W_{K,U} = \{f \in C^\infty(M,N) : f(K) \subset U\}$. The C^0 *topology* or *compact open topology* on $C^\infty(M,N)$ has

$$\{W_{K,U} : \text{compact } K \subset M, \text{open } U \subset N\}$$

as a sub-base.

Definition *C^1-topology* on $C^\infty(M,N)$ is the C^0 topology on $C^\infty(TM,TN)$. Note C^1 topology has more open sets, which are necessary to satisfy condition (2). More explicitly, for $f_0 \in C^\infty(M,N)$, charts $\phi : U \to \mathbf{R}^m$, $\psi : V \to \mathbf{R}^n$ for M, N respectively, and compact $K \subset U$ and $\epsilon > 0$, define

$$W_{f_0,\phi,\psi,K,\epsilon} = \{f \in C^\infty(M,N) : f(K) \subseteq V \text{ and } \forall x \in K \|(D\psi f \phi^{-1})_{\phi(x)} - (D\psi f_0 \phi^{-1})_{\phi(x)}\| < \epsilon\}$$

(will be unhappy unless $f_0(K) \subseteq V$). $\mathcal{W} \subset C^\infty(M,N)$ is *C^1-open* iff $\forall f_0 \in \mathcal{W} \exists \phi_i, \psi_i, K_i, \epsilon_i$ $(1 \leq i \leq r)$ such that $f_0 \in \bigcap_{i=0}^r W_{f_0,\phi_i,\psi_i,\epsilon_i} \subseteq \mathcal{W}$.

C^0-open $\Longrightarrow$ C^1-open. Need to check following Lemmas:

Lemma *Suppose $P^p \subset N^n$ is a closed C^∞ submanifold, and $A \subset M^m$ compact. Then $\{f \in C^\infty(M,N) : f \pitchfork_A P\}$ is C^1-open in $C^\infty(M,N)$.*

Lemma *If $f : M \times T \to N$ is a C^∞ map then $\hat{f} : T \to C^\infty(M,N)$ defined by $\hat{f}(t)(x) = f(x,t) = f_t(x)$ is continuous with respect to the C^1 topology on $C^\infty(M,N)$.*