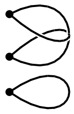
Reference Finite Common Covers of Graphs by Frank Leighton in the Journal of Combinatorial Theory, Series B 33, pages 231–238, 1982.

Definition Γ *covers* another graph G if $\gamma : \Gamma \to G$ is node, edge, and incidence preserving, onto, and locally one-to-one.

Note that loops and multi-edges are allowed. Loops are counted as being incident to a vertex twice. Edges are not considered oriented or having a direction. This definition of graph cover can be stated completely combinatorially but agrees with the topological definition.

Example 

Definition A *Universal Cover* $U(G)$ is the (usually infinite) tree which covers G.

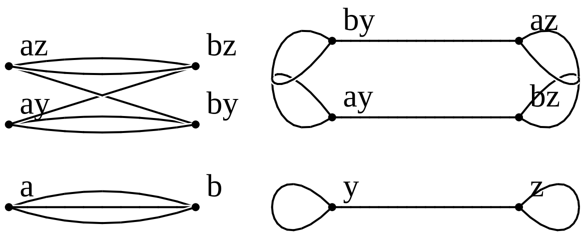
Fact 1 The universal cover covers any (connected) graph which covers G. It is unique up to isomorphism.

Main Theorem If G and G' are finite, connected graphs, then the following are equivalent:

1. G and G' share a common finite cover,
2. G and G' have the same universal cover,
3. G and G' share a common (possibly infinite) cover,
4. G and G' have the same degree refinement.

The goal is to get (2) implies (1), but (1) $\Longrightarrow$ (2) $\Longrightarrow$ (3) $\Longrightarrow$ (4) are straightforward from a few facts.

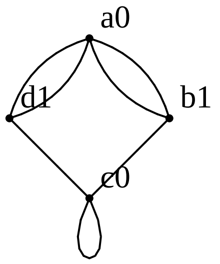
Example Lips and Barbell have the infinite 3-valent tree as a universal cover and have a common double cover.

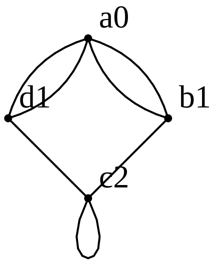


Fact 2 If Γ covers H via σ, and H covers G via γ, then Γ covers G via $\gamma\sigma$.

Definition A *degree partition* of a graph G is a minimal way of dividing the vertices of G into blocks satisfying the condition that there are constants r_{ij} which are the number of edges from each vertex in block i to a vertex in block j. The matrix (r_{ij}) is called a *degree refinement*. Note that a loop counts twice in r_{ii} and that $\sum_j r_{ij}$ is the degree of every vertex in block i.

Algorithm Computing a degree partition: First divide the vertices into blocks according to their degree. Subdivide blocks if vertices within a block would induce different constants r_{ij} according to the current partition. Repeat until the matrix r_{ij} is well defined.

Example 



(r_{ij}) =— INSERT MATRIX —

Fact 3 If Γ covers H via σ, then Γ and H have the same degree refinement. In fact, σ preserves the degree partition.

Fact 4 Two graphs with the same degree refinement have the same universal cover.

Why? Start anywhere and construct the universal cover recursively directly from the degree refinement information.

On with the proof. We're going to replace all the edges with two directed edges oriented oppositely.

Definition Vertices are of type i if they are in the i^{th} block.

Definition Edges are of type $i \to j$ if they go from a vertex of type i to one of type j. Note that we are using the fact that edges are oriented.

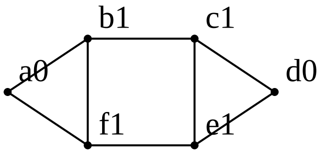
Let n_i be the number of type i vertices in G.

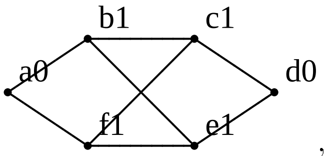
Let m_{ij} be the number of type $i \to j$ vertices in G.

Fact $n_i r_{ij} = m_{ij}$ and $m_{ij} = m_{ji}$.

Let $s = lcm_{i,j}(m_{ij})$, $a_i = s/n_i$, and $b_{ij} = s/m_{ij}$.

a_i and b_{ij} are necessary to get the ratios of the vertices correct in the cover we are constructing: for every pair of vertices $v \in G$, $v' \in G'$ of the same type i, will have a_i vertices in the cover. Similarly for pairs of edges $e \in G$, $e' \in G'$ of type $i \to j$ and b_{ij}.

Example 

and 

, $s = 8$, $a_0 = 4$, $a_1 = 2$, compensates for there being 2 type 0s and 4 type 4s in base.

Degree refinements dictate the ratio of vertices of different types. r_{ij}/r_{ji} is the ratio of the number of j type vertices to the number of i type vertices. This in fact implies a_i and b_{ij} are functions solely of r_{ij}

Picture of typical type i nodes of G, G', and H — Picture —.

The actual edge relation relies on choosing a linear ordering of the edges from a particular vertex.

The formula for a reverse of a given edge: $(i, j, e, e', q)^r = (j, i, e^r, e'^r, q)$. Covering map is: $(i, v, v', p) \mapsto v \in G$ and $(i, v, v', p) \mapsto v' \in G'$, and similarly for edges.

The cover constructed may not be connected, but each connected component is a cover of both G and G'.