Combinatorics Study Guide

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Counting

Definition If $f : \mathbf{N} \to \mathbf{C}$, then the *ordinary generating function* of $f(n)$ is $\sum_{n\geq 0} f(n)x^n$. The *exponential generating function* of $f(n)$ is $\sum_{n\geq 0} f(n)x^n/n!$. Ignore questions of convergence.

Definition If $F(x) = \sum_{n\geq 0} a_n x^n$, we call a_n the *coefficient* of x^n in $F(x)$ and write $a_n = F(X)|_n$.

$$\left(\sum_{n\geq 0} a_n x^n\right)\left(\sum_{n\geq 0} b_n x^n\right) = \sum_{n\geq 0} c_n x^n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

$$\left(\sum_{n\geq 0} \frac{a_n x^n}{n!}\right)\left(\sum_{n\geq 0} \frac{b_n x^n}{n!}\right) = \sum_{n\geq 0} \frac{d_n x^n}{n!} \text{ where } d_n = \sum_{i=0}^{n} \binom{n}{i} a_i b_{n-i}$$

$$\left(\sum_{n\geq 0} \alpha^n x^n\right)(1-\alpha x) = 1$$

Definition If $F_1(x), F_2(x), \ldots$ is a sequence of formal power series, and if $F(x) = \sum_{n\geq 0} a_n x^n$ is another power series, we say that $F_i(x)$ *converges* to $F(x)$ as $i \to \infty$ provided that for all $n \geq 0$ there is a number $\delta(n)$ such that the coeffficients of x^n in $F_i(x)$ is a_n whenever $i \geq \delta(n)$.

Definition Define the *degree* of a non-zero formal power series $F(x) = \sum_{n\geq 0} a_n x^n$ denoted $\deg F(x)$, to be the least integer n such that $a_n \neq 0$.

Proposition *The infinite series $\sum_{j\geq 0} F_j(x)$ converges if and only if $\lim_{j\to\infty} \deg F_j(x) = \infty$.*

Proposition *The infinite product $\prod_{j\geq 1}(1 + F_j(x))$ where $F_j(0) = 0$ converges if and only if $\lim_{j\to\infty} \deg F_j(x) = \infty$.*

Definition If $F(x) = \sum_{n\geq 0} a_n x^n$ and $G(x)$ are formal power series with $G(0) = 0$, define the *composition* $F(G(x))$ to be the infinite sum $\sum_{n\geq 0} a_n G(x)^n$ (well-defined).

Example If $F(x)$ is a formal power series with $F(0) = 0$, then we can *define* for any $\lambda \in \mathbf{C}$ the formal power series

$$(1 + F(x))^\lambda = \sum_{n\geq 0} \binom{\lambda}{n} F(x)^n$$

where $\binom{\lambda}{n} = \lambda(\lambda-1)\cdots(\lambda-n+1)/n!$.

Definition If $F(x) = \sum_{n\geq 0} a_n x^n$, define the *formal derivative* $F'(x)$ to be the formal power series $\sum_{n\geq 0}(n+1)a_{n+1}x^n$.

Example The generating function $F(x) = \sum_{n\geq 0} a_n x^n$ with $a_0 = a_1 = 1$, and $a_n = a_{n-1} + a_{n-2}$ if $n \geq 2$ satisfies

$$F(x) = 1 + x + x(F(x) - 1) + x^2 F(x)$$

so solving for $F(x)$ yields $F(x) = 1/(1-x-x^2)$.

Sets, multisets

Definition Let $\binom{S}{k}$ to be the set of all k-element subsets of S, and $\binom{n}{k} = \#\binom{S}{k}$ for S any n element set.

Definition Let $(n)_k$ be the number of ways of choosing a k-subset T of S and then linearly ordering the elements of T. Then

$$(n)_k = \binom{n}{k} k! = n(n-1)\cdots(n-k+1)$$

$$(1+x)^n = \sum_{k\geq 0} \binom{n}{k} x^k$$

Definition A *composition* of N is an expression of n as an *ordered* sum of positive integers. For instance, there are eight compositions of 4; namely,

$$\begin{array}{llll} 1+1+1+1 & 2+1+1 & 1+2+1 & 1+1+2 \\ 3+1 & 1+3 & 2+2 & 4 \end{array}$$

If exactly k summands appear in a composition σ, we say that σ has k *parts*, and we call σ a *k-composition*. There are $\binom{n-1}{k-1}$ k-compositions of n and 2^{n-1} compositions of n (stars and bars argument).

Definition A *weak k-composition of* n is a solution to $x_1 + x_2 + \cdots + x_k = n$. These are counted by adding 1 to each x_i and then adding k to the right hand side, so there are $\binom{n+k-1}{k-1}$ weak k-compositions of n.

Definition A *multiset* is a set with repeated elements, represented by a function $S \to \mathbf{N}$. There are $\left(\binom{n}{k}\right) = \binom{n+k-1}{k}$ multisets with k elements of an n element set.

Permutations

Permutations $\mathcal{S}_n = \{\pi : [n] \to [n]\}$ may be written as a disjoint union of cycles. Define the *standard representation* by requiring that each cycle is written with its largest element first and the cycles are written in increasing order of their largest element. This defines a bijection from $\mathcal{S}_n$ to itself by 'removing the parenthesis' from π and then considering $\hat{\pi}(k)$ to be the k^{th} number written. The inverse map is defined by re-inserting parentheses at ever *left-to-right maximum.*

Definition Let the *type* of π be the sequence $(c_1, \ldots, c_n)$ where $c_i = c_i(\pi)$ is the number of cycles of length i.

Proposition *The number of $\pi \in \mathcal{S}(S)$ of type $(c_1, \ldots, c_n)$ is equal to*

$$\frac{n!}{1^{c_1} c_1! 2^{c_2} c_2! \cdots n^{c_n} c_n!}\,.$$

Definition Let $c(n,k)$ to be the number of $\pi \in \mathcal{S}_n$ with exactly k cycles. $s(n,k) = (-1)^{n-k} c(n,k)$ is known as a *Stirling number of the first kind.*

Lemma *The numbers $c(n,k)$ satisfy the recurrence*

$$c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$$

for $n, k \geq 1$ and $c(n,k) = 0$ if $n \leq 0$ or $k \leq 0$, except $c(0,0) = 1$.

Proof Divide permutations in $\mathcal{S}_n$ into those that fix n and those that don't. $\surd$

Proposition *For x an indeterminate and fixed $n \geq 0$,*

$$\sum_{k=0}^{n} c(n,k) x^k = x(x+1)(x+2)\cdots(x+n-1)$$

Definition Given a permutation $\pi = b_1 b_2 \cdots b_n$, then a pair (b_i, b_j) is called an *inversion* of π if $i < j$ and $b_i > b_j$.

Proposition *Let $i(\pi)$ denote the number of inversion of the permutation $\pi \in \mathcal{S}_n$. Then*

$$\sum_{\pi \in \mathcal{S}_n} q^{i(\pi)} = (1+q)(1+q+q^2)\cdots(1+q+q^2+\cdots+q^{n-1})\,.$$

Definition If $\pi = a_1 a_2 \cdots a_n \in \mathcal{S}_n$, then define the *descent set* to be $D(\pi) = \{i | a_i > a_{i+1}\}$.

The Twelvefold Way

Consider finite sets N and X, with $|N| = n$ and $|X| = x$. There are three types of restrictions on functions $f : N \to X$:

a. f is arbitrary (no restriction)

b. f is injective (one-to-one)

c. f is surjective (onto)

When counting such functions f, we consider equivalence classes defined by whether the elements of N are distinguishable and whether the elements of X are distinguishable. If the elements of N are not distinquished, then we define an equivalence relation $f \sim g$ if there is a bijection $\pi : N \to N$ such that $f \circ \pi = g$.

Elements of N	**Elements of X**	**Any f**	**Injective f**	**Surjective f**
dist.	dist.	1. x^n	2. $(x)_n$	3. $x! S(n,x)$
indist.	dist.	4. $\left(\!\binom{x}{n}\!\right)$	5. $\binom{x}{n}$	6. $\left(\!\binom{x}{n-x}\!\right)$
dist.	indist.	7. $\sum_{k=1}^{x} S(n,k)$	8. 1 if $n \leq x$	9. $S(n,x)$
indist.	indist.	10. $\sum_{k=1}^{x} p_k(n)$	11. 1 if $n \leq x$	12. $p_x(n)$

6. Since the N are indistinguishable, we only care about how many elements of N map to each $b \in X$. Since f is surjective, this number is at least 1, so subtract 1 element of N for each element of X to reduce it to case 4.

9. A *partition* of a finite set N is a collection $\pi = \{B_1, B_2, \ldots, B_k\}$ of subsets of N such that $B_i \neq \emptyset$ for each i, $B_i \cap B_j = \emptyset$ if $i \neq j$, and $\bigcup_i B_i = N$. We call B_i a *block* of π, and say that π has k blocks, denoted $|\pi| = k$. Define $S(n,k)$ to be the number of partitions of an n-set into k blocks. $S(n,k)$ is called a *Stirling number of the second kind.* By convention, set $S(0,0) = 1$. Note for $n \geq 1$, $S(n,k) = 0$ if $k > n$, $S(n,0) = 0$, $S(n,1) = 1$, $S(n,2) = 2^{n-1} - 1$, $S(n,n) = 1$, $S(n,n-1) = \binom{n}{2}$. They satisfy the following basic recurrence:

$$S(n,k) = kS(n-1,k) + S(n-1,k-1)$$

(given a partition of $n-1$ can get a partition of n either by adding n to any of the k existing blocks, or putting it in a block by itself). Also note that $x^n = \sum_{k=0}^{n} S(n,k)(x)_k$. So if f is surjective and we are not distinguishing the elements of X, then none of the sets $f^{-1}(b)$ are void, so we may bijectively associate a partition with such f.

3. Same as 9 but there are $x!$ ways of specifying which block corresponds to each element of x.

7. Again the same as 9, except we consider separately f with k elements in its image. For each k, f is surjective onto its image, so and this is counted by 9 above.

12. Define a *partition* of $n \in \mathbf{N}$ to be a sequence $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbf{N}^k$ such that $\sum \lambda_i = n$ and $\lambda_1 \geq \cdots \geq \lambda_k$. We may also informally regard a partition as a way of writing n as a sum $\lambda_1 + \cdots + \lambda_k$ of positive integers, *disregarding the order of summands.* If λ is a partition of n, we write $\lambda \vdash n$. The nonzero terms are called *parts* of λ and we say that λ has k parts. If the partition λ has α_i parts equal to i, then we write $\lambda = \langle 1^{\alpha_1}, 2^{\alpha_2}, \ldots \rangle$. We write $p(n)$ for the total number of partitions of n, and $p_k(n)$ for the number of partitions of n with exactly k parts. It is easy to verify the recurrence $p_k(n) = p_{k-1}(n-1) + p_k(n-k)$ (either $\lambda \vdash n$ has last part equal to 1 (drop it), or it doesn't (drop one from every part)). $p(n)$ has the generating function

$$\sum_{n \geq 0} p(n) x^n = \prod_{i \geq 1} (1 - x^i)^{-1}$$

(the number of parts λ has equal to i determines which term of $(1-x^i)^{-1} = 1 + x^i + x^{2i} + \cdots$ λ corresponds to).

Now if $f : N \to X$ is surjective, then f corresponds bijectively to the partition $\lambda = (|f^{-1}(b_1)|, \ldots, |f^{-1}(b_x)|)$ with exactly x non-zero parts.

10. Same argument as 7.

Inclusion-Exclusion

Theorem *Let S be an n-set. Let V be the 2^n-dimensional vector space (over some field k) of all functions $f : 2^S \to k$. Let $\phi : V \to V$ be the linear transformation defined by*

$$\phi f(T) = \sum_{Y \supseteq T} f(Y), \text{ for all } T \subseteq S.$$

Then ϕ^{-1} exists and is given by

$$\phi^{-1}f(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f(Y), \text{ for all } T \subseteq S.$$

We think of S as being a set of properties that elements of some given set A of objects may or may not have. For any subset T of S, let $f_=(T)$ be the number of objects in A that have *exactly* the properties in T (so they do not have the properties in $T^c = S - T$). Let $f_\geq(T)$ be the number of objects in A that have *at least* the properties in T. Then

$$f_\geq(T) = \sum_{Y \supseteq T} f_=(Y).$$

By the theorem,

$$f_=(T) = \sum_{Y \supseteq T} (-1)^{|Y-T|} f_\geq(Y).$$

In particular, the number of objects having *none* of the properties in S is given by

$$f_=(\emptyset) = \sum_Y (-1)^{|Y|} f_\geq(Y),$$

where Y ranges over all subsets of S.

One thinks of $f_\geq(T)$ as being a first approximation to $f_=(T)$. Then subtract

$$\sum_{Y \supseteq T, |Y-T|=1} f_\geq(Y).$$

to get a better approximation, add back in

$$\sum_{Y \supseteq T, |Y-T|=2} f_\geq(Y).$$

and so on.

The standard formulation of inclusion-exclusion, one just considers subsets of A. Let $A_1, \ldots, A_n$ be subsets of a finite set A. For each subset T of $[n]$, let $A_T = \bigcap_{i \in T} A_i$, (with $A_\emptyset = A$) and for $0 \leq k \leq n$ set $S_k = \sum_{|T|=k} |A_T|$, the sum of the cardinalities of all k-tuple intersection of the A_i's. Think of A_i as defining a property P_i by the condition that $x \in A$ satisfies P_i if and only if $x \in A_i$. Then A_T is just the set of objects in A that have at least the properties in T. Threfore the number of elements of A lying in *none* of the A_i's is given by

$$\#(A_1^c \cap \cdots \cap A_n^c) = S_0 - S_1 + S_2 - \cdots + (-1)^n S_n,$$

where $S_0 = |A_\emptyset| = |A|$.

The Principle of Inclusion-Exclusion may be dualized by interchanging $\cap$ with $\cup$, $\subseteq$ with $\supseteq$, and so on, throughout. If we let $f_\leq(T)$ be the number of objects of A having *at most* the properties in T, then

$$f_\leq(T) = \sum_{Y \subseteq T} f_=(Y), \quad f_=(T) = \sum_{Y \subseteq T} (-1)^|T - Y| f_\leq(Y).$$

A common special case occurs when the function $f_=$ satisfies $f_=(T) = f_=(T')$ whenever $|T| = |T'|$ (and thus $f_\geq(T)$ only depends upon the size of T). We set $a(n-i) = f_=(T)$ and $b(n-i) = f_\geq(T)$ whenever $|T| = i$. We thus obtain the formulas

$$\begin{aligned} b(m) &= \textstyle\sum_{i=0}^m \binom{m}{i} a(i), & 0 \leq m\,, \\ a(m) &= \textstyle\sum_{i=0}^m \binom{m}{i} (-1)^{m-i} b(i), & 0 \leq m\,. \end{aligned}$$

Example Derangement problem: the number of permutations with no fixed points. Let A_i be the set of π which fix i. The number of permutations fixing at least the set T is $f_\geq(T) = b(n-i) = (n-i)!$ where $|T| = i$. Therefore the number $f_=(\emptyset) = a(n) = D(n)$ of permutations with *no* fixed points is

$$D(n) = \sum_{i=0}^{n} \binom{n}{i} (-1)^{n-i} i!\,.$$

Partially Ordered Sets

Definition A *partially ordered set* P, or *poset*, is a set together with a binary relation denoted $\leq$ satisfying reflexivity, antisymmetry ($x \leq y$ and $y \leq x$ implies $x = y$), and transitivity.

Definition We say two elements x and y of P are *comparable* if $x \leq y$ or $y \leq x$.

Definition chain

Definition antichain

Definition multichain

Lattices

Definition An *upper bound* of two elements x and y of P is an element $z \in P$ such that $z \geq x$ and $z \geq y$. A *least upper bound* of x and y is an upper bound z such that every upper bound w satisfies $w \geq z$. If a least upper bound exists it is unique and denoted $x \vee y$ in a counter-intuitive fashion, read "x join y" or "x sup y". Similarly define the greates lower bound $x \wedge y$, read "x meet y" or "x inf y."

Definition A *lattice* is a poset L such that ever pair of elements has a least upper bound and greatest lower bound. In a lattice L: the operations $\vee$ and $\wedge$ are associative, commutative, and idempotent ($x \wedge x = x \vee x = x$); $x \wedge (x \vee y) = x = x \vee (x \wedge y)$ (absorption laws); $x \wedge y = x \iff x \vee y = y \iff x \leq y$.

Clearly all finite lattices have a $\hat{0}$ and $\hat{1}$. If L and M are lattices, then so are L^*, $L \times M$, and $L \oplus M$. However $L + M$ will never be a lattice unless one of L or M is void, but $\widehat{L + M}$ is always a lattice.

Definition If every pair of elements of a poset P has a meet (respectively, join), we say that P is a *meet-semilattice* (respectively *join-semilattice*).

Proposition *Let P be a finite meet-semilattice with $\hat{1}$. Then P is a lattice. Dually, a finite join-semilattice with $\hat{0}$ is a lattice.*

Proof Page 103. $\surd$

Proposition *Let L be a finite lattice. The following two conditions are equivalent:*

i. L is graded, and the rank function ρ of L satisfies $\rho(x)+\rho(y)\geq\rho(x\wedge y)+\rho(x\vee y)$ for all $x,y\in L$.

ii. If x and y both cover $x\wedge y$, then $x\vee y$ covers both x and y.

Proof Pages 103–104. $\surd$

Definition finite upper semimodular lattice **Definition** modular lattice etc.

Distributive Lattices

Definition A *distributive lattice* is a lattice which for all elements x,y,z satisfies the distributive laws

$$\begin{aligned} x\vee(y\wedge z) &= (x\vee y)\wedge(x\vee z) \\ x\wedge(y\vee z) &= (x\wedge y)\vee(x\wedge z)\,. \end{aligned}$$

Every distributive lattice is modular. $J(P)$ the set of order ideals of P ordered by inclusion, is a distributive lattice. The lattice operations $\vee$ and $\wedge$ on order ideals are just ordinary union and intersection (as subsets of P).

Fundamental Theorem of Finite Distributive Lattices (FTFDL) *Let L be a finitie distributive lattice. Then there is a unique (up to isomorphism) finite poset P for which $L\cong J(P)$.*

Incidence Algebra

Let P be a locally finite poset, and let $\mathrm{Int}(P)$ denote the set of intervals of P (Recall that the void set is not an interval). Let K be a field. If $f:\mathrm{Int}(P)\to K$, then write $f(x,y)$ for $f([x,y])$.

Definition The *incidence algebra* $I(P,K)$ of P over K is the K-algebra of all functions $f:\mathrm{Int}(P)\to K$ (with the usual structure of a vector space over K), where multiplication (or *convolution*) is defined by

$$(fg)(x,y)=\sum_{x\leq z\leq y} f(x,z)g(z,y)\,.$$

The sum is defined since P is locally finite. For us, K will always be $\mathbf{C}$, and we will write simply $I(P)$ for $I(P,\mathbf{C})$. This is an associative K-algebra with (two-sided) identity denoted δ or 1, defined by

$$\delta(x,y)=\begin{cases}1, & \text{if } x=y\\ 0, & \text{if } x\neq y\,.\end{cases}$$

Proposition *Let $f\in I(P)$. The following conditions are equivalent.*

a. f has a left inverse.

b. f has a right inverse.

c. f has a two-sided inverse.

d. $f(x,x)\neq 0$ for all $x\in P$.

Moreover, if f^{-1} exists, then $f^{-1}(x,y)$ depends only on the poset $[x,y]$.

Definition The *zeta function* ζ is defined by $\zeta(x,y)=1$ for all $x\leq y$ in P. Thus

$$\zeta^2(x,y)=\sum_{x\leq z\leq y}1=\mathrm{card}[x,y]$$

More generally, if $k\in\mathbf{P}$ then

$$\zeta^2(x,y)=\sum_{x=x_0\leq x_1\leq\cdots\leq x_k=y}1\,,$$

the number of multichains of length k from x to y. Similarly,

$$(\zeta-1)(x,y)=\begin{cases}1, & \text{if } x<y\\ 0, & \text{if } x=y\,.\end{cases}$$

Hence if $k\in\mathbf{P}$ then $(\zeta-1)^k(x,y)$ is the number of chains $x=x_0<x_1<\cdots<x_k=y$ of length k from x to y.

Check 3.4.1 and 3.5.2 to see stuff about chains in a distributive lattice.

$$(2-\zeta)=\begin{cases}1, & \text{if } x=y\\ -1, & \text{if } x<y\ .\end{cases}$$

By the proposition above, $2-\zeta$ is invertible. $(2-\zeta)^{-1}(x,y)$ is the *total* number of chains $x=x_0<x_1<\cdots<x_k=y$ from x to y. Why? Observe

$$2-\zeta=\frac{1}{1-(\zeta-1)}=1+(\zeta-1)+(\zeta-1)^2+\cdots$$

This is justified by restricting to an interval, so the right hand side is a finite sum (ending at $(\zeta-1)^l$ the longest chain in $[x,y]$), and then multiplying through by the denominator.

Möbius Inversion

Definition It follows from the above proposition that the zeta function ζ of a locally finite posit P is invertible; its inverse is called the *Möbius function* of P and is denoted μ. One can define μ inductively without reference to the incidence algebra. Namely, the relation $\mu\zeta=\delta$ is equivalent to

$$\begin{aligned}\mu(x,x)&=1, \quad \text{for all } x\in P\\ \mu(x,y)&=-\sum_{x\leq z<y}\mu(x,z), \quad \text{for all } x<y \text{ in } P.\end{aligned}$$

Möbius Inversion Formula *Let P be a poset in which every principal dual order ideal is finite. Let $f,g:P\to\mathbf{C}$. Then*

$$g(x)=\sum_{y\geq x}f(y), \quad \textit{for all } x\in P,$$

if and only if

$$f(x)=\sum_{y\geq x}g(y)\mu(x,y) \quad \textit{for all } x\in P.$$

Proof The set $\mathbf{C}^p$ of all functions $P \to \mathbf{C}$ forms a vector space on which $I(P)$ acts (on the right) as an algebra of linear transformations by

$$(f\xi)(x) = \sum_{y \leq x} \xi(y, x)$$

for $f \in \mathbf{C}^p$ and $\xi \in I(P)$. $\surd$

(Dual Form) *Let P be a poset in which every principal order ideal is finite. Let $f, g : P \to \mathbf{C}$. Then*

$$g(x) = \sum_{y \leq x} f(y), \quad \text{for all } x \in P,$$

if and only if

$$f(x) = \sum_{y \leq x} g(y)\mu(y, x) \quad \text{for all } x \in P.$$

The Product Theorem *Let P and Q be locally finite posets. If $(x, y) \leq (x', y')$, then $\mu_{P\times Q}((x, y), (x', y')) = \mu_P(x, y)\mu_Q(y, y')$.*

Proof Compute the sum over an interval, check that its 1 if and only if the interval is a single element. $\surd$

Example Applying Product Theorem to $B_n \cong \mathbf{2}^n$, gives μ_{B_n}. Then the Möbius inversion formula is exactly the Principle of Inclusion-Exclusion.

Example On D_N the Möbius function $\mu(r, s)$ corresponds exactly to the classical number-theoretic Möbius function $\mu(s/r)$.

Proposition *Let P be a finite poset and let $\hat{P}$ denote P with $\hat{0}$ and $\hat{1}$ adjoined. Let c_i be the number of chains $\hat{0} = x_0 < x_1 < \cdots < x_i = \hat{1}$ of length i between $\hat{0}$ and $\hat{1}$ (so $c_0 = 0$ and $c_1 = 1$). Then $\mu_{\hat{P}}(\hat{0}, \hat{1}) = c_0 - c_1 + c_2 - \cdots$.*

Proof $\mu = (1 + (\zeta - 1))^{-1}$ $\surd$

Definition Define the *order complex* of P to be the simplicial complex $\Delta(P)$ with verticies of $\Delta(P)$ the elements of P, and faces of $\Delta(P)$ being chains in P.

Proposition *For P a finite poset, the above proposition is equivalent to $\mu_{\hat{P}}(\hat{0}, \hat{1}) = \tilde{\chi}(\Delta(P))$, for $\tilde{\chi}$ the reduced Euler characteristic $\tilde{\chi} = \chi - 1$.*

Corollary *In any locally finite poset P, $\mu_P(x, y) = \mu_{P^*}(y, x)$.*

Definition If a finite graded poset P with $\hat{0}$ and $\hat{1}$ satisfies $\mu_P(x, y) = (-1)^{l(x,y)}$ whenever $(x, y) \neq (\hat{0}, \hat{1})$, then P is said to be *semi-Eulerian*. If in addition $\mu_P(\hat{0}, \hat{1}) = (-1)^{l(\hat{0},\hat{1})}$, then P is called *Eulerian*.

Lattice Möbius Algebras

Definition Let L be a lattice and K a field. The *Möbius algebra* $A(L, K)$ is the semigroup algebra of L with the meet operation, over K. In other words, $A(L, K)$ is the vector space over K of formal linear combinations of elements of L, with (bilinear) multiplication defined by $x \cdot y = x \wedge y$ for all $x, y \in L$.

There is an isomorophism between $A(L, K)$ and the product of $|L|$ copies of K defined by sending $x \in L$ to the identity in K_x.

Corollary *Let L be a finite lattice with at least two elements, and let $\hat{1} \neq a \in L$. Then*

$$\sum_{x : x \wedge a = \hat{0}} \mu(x, \hat{1}) = 0\,.$$

Corollary *Let L be a finite lattice, and let X be a subset of L such that $\hat{1} \notin X$, and if $y \in L$ and $y \neq \hat{1}$, then $y \leq x$ for some $x \in X$. Then $\mu(\hat{0}, \hat{1}) = \sum_k (-1)^k N_k$, where N_k is the number of k-subsets of X whose meet is $\hat{0}$.*

Corollary *If L iis a finite lattice for which $\hat{0}$ is not a meet of coatoms, then $\mu(\hat{0}, \hat{1}) = 0$. Dually, if $\hat{1}$ is not a join of atoms, then again $\mu(\hat{0}, \hat{1})$.*

Example Let $L = J(P)$ be a finite distributive lattice. Then

$$\mu(I, I') = \begin{cases} (-1)^{l(I,I')} = (-1)^{|I'-I|} & \text{if } I' - I \text{ is an antichain of } P \\ 0 & \text{otherwise.} \end{cases}$$

Möbius function on Semimodular Lattices

Proposition *The Möbius function of a finite semimodular lattice alternates in sign.*

Example Partitions of a finite set S ordered by refinement.

Zeta Polynomials

Definition Let P be a finite poset. If $n \geq 2$, then define $Z(P, n)$, the *zeta polynomial* of P, to be the number of multichains $x_1 \leq x_2 \leq \cdots \leq x_{n-1}$ in P.

a. *Let b_i be the number of chains $x_1 < x_2 < \cdots < x_{i-1}$ in P. Then*

$$Z(P, n) = \sum_{i \geq 2} b_i \binom{n-2}{i-2}\,.$$

So $Z(P, n)$ is a polynomial function of n, whose degree is the length of the longest chain of P. Also, $Z(P, 2) = |P|$.

b. *Since $Z(P, n)$ is a polynomial for all integers $n \geq 2$, we can define it for $n \in \mathbf{Z}$. Then $Z(P, 1) = 1 + \mu_{\hat{P}}(\hat{0}, \hat{1})$.*

c. *If P has a $\hat{0}$ and $\hat{1}$, then $Z(P, n) = \zeta^n(\hat{0}, \hat{1})$ for all $n \in \mathbf{Z}$. In particular, $Z(P, -1) = \mu(\hat{0}, \hat{1})$, $Z(P, 0) = 0$ if $\hat{0} \neq \hat{1}$, and $Z(P, 1) = 1$.*

Definition If $m \in \mathbf{P}$, then let $\Omega(P, m)$ denote the number of order-preserving maps $\sigma : P \to \mathbf{m}$. Note $\Omega(P, m) = Z(J(P), m)$. Hence $\Omega(P, m)$ is a polynomial function of m of degree $|P|$. $\Omega(P, m)$ is called the *order polynomial* of P.

Rank-selection

Let P be a finite graded poset of rank n, with rank function $\rho : P \to [0, n]$. If $S \subseteq [0, n]$ then define the subposet

$$P_S = \{x \in P : \rho(x) \in S\}$$

called the S*-rank-selected subposet* of P. For instance, $P_\emptyset = \emptyset$ and $P_{[0,n]} = P$. Now define $\alpha(P, S) = \alpha(S)$ to be the number of maximal chains of P_S. For instance $\alpha(\{i\}) = \alpha(i)$ is the number of elements of P rank i. Define $\beta(P, S) = \beta(S)$ by

$$\beta(S) = \sum_{T \subseteq S} (-1)^{|S-T|} \alpha(T) .$$

Equivalently, by Inclusion-Exclusion,

$$\alpha(S) = \sum_{T \subseteq S} \beta(T) .$$

If μ_S denotes the Möbius function of the poset $\hat{P}_S$, then

$$\beta(S) = (-1)^{|S|-1} \mu_S(\hat{0}, \hat{1}) .$$

We call β the *rank-selected Möbius invariant of* P.

Suppose P has a $\hat{0}$ and $\hat{1}$. Then

$$\begin{aligned} \alpha(P, S) &= \alpha(P, S \cap [n-1]) \\ \beta(P, S) &= 0 \text{ if } S \not\subseteq [n-1] \text{ (i.e., if } 0 \in S \text{ or } n \in S) \end{aligned}$$

So we lose nothing restricting $S \subseteq [n-1]$. So if P is a lattice, we will only consider such S.

For some P, we can hope to give combinatorial meaning to α and β. Consider $L = J(P)$ a finite distributive lattice of rank n (so $|P| = n$). Regard P as a partial ordering of the set $[n]$, and assume that P is compatible with the usual ordering of $[n]$; that is if $i < j$ in P then $i < j$ in $\mathbf{Z}$. We then call P a *natural* partial order on $[n]$. We may identify an extension $\sigma : P \to [n]$ of P to a total order with a permutation $\sigma^{-1}(1), \ldots, \sigma^{-1}(n)$ of $[n]$. The set of all $e(P)$ permutations of $[n]$ obtained in this way is denoted $\mathcal{L}(P)$ and is called the *Jordan-Hölder set* of P. Now given L $(= J(P))$, and $S \subseteq [n-1]$, then $\beta(L, S)$ is equal to the number of permutations $\pi \in (P)$ with descent set S.

Proof Show a bijection between chains in $\alpha(L, S)$ and permutations $\pi \in \mathcal{L}(P)$ whose descent set is contained in S. Note that such chains in L are nested chains of order ideals, and arranging $I_{i+1} - I_i$ in increasing order gives the bijection. $\surd$

Eulerian Posets

Proposition *Let P be Eulerian of rank n. Then $Z(P, -m) = (-1)^n Z(P, m)$.*

Lemma *Let P be a finite poset with $\hat{0}$ and $\hat{1}$, and let $x \in P - \{\hat{0}, \hat{1}\}$. Then*

$$\mu_{P-x}(\hat{0}, \hat{1}) = \mu_P(\hat{0}, \hat{1}) - \mu_P(\hat{0}, x) \mu_P(x, hat1) .$$

Proof Use the theorem that says $\mu(x, y)$ is the alternating sum of the lengths of the chains from x to y. $\surd$

Lemma *Let P be as above, and let Q be any subposet of P containing $\hat{0}$ and $\hat{1}$. Then*

$$\mu_Q(\hat{0}, \hat{1}) = \sum (-1)^k \mu_P(\hat{0}, x_1) \mu_P(x_1, x_2) \cdots \mu_P(x_k, 1) ,$$

where the sum ranges over all chains $\hat{0} < x_1 < \cdots < x_k < \hat{1}$ *in* P *such that* $x_i \notin Q$ *for all* i.

Proposition *Let* P *be Eulerian of rank* n, *and let* Q *be any subposet of* P *containing* $\hat{0}$ *and* $\hat{1}$. *Set* $\bar{Q} = (P - Q) \cup \{\hat{0}, \hat{1}\}$. *Then*

$$\mu_Q(\hat{0}, \hat{1}) = (-1)^{n-1} \mu_{\bar{Q}}(\hat{0}, \hat{1}) .$$

Corollary *Let* P *be Eulerian of rank* n, *let* $S \subseteq [n-1]$, *and set* $\bar{S} = [n-1] - S$. *Then* $\beta(P, S) = \beta(P, \bar{S})$.

Binomial Posets and Generating Functions

Definition A poset P is called a *binomial poset* if it satisfies the three conditions:

a. P is locally finite with $\hat{0}$ and contains an infinite chain.

b. Every interval $[x, y]$ of P is graded. If $l(x, y) = n$, then we call $[x, y]$ an n-interval.

c. For all $n \in \mathbf{N}$, any two n-intervals contain the same number $B(n)$ of maximal chains. We call $B(n)$ the *factorial function* of P.

From the definition, $B(0) = B(1) = 1$, $B(2) = \mathrm{card}(x, y)$ where $[x, y]$ is any 2-interval, and $B(0) \leq B(1) \leq B(2) \leq \cdots$.

Example Let $P = \mathbf{N}$ with the usual linear order. Then $B(n) = 1$ for all $n \in \mathbf{N}$.

Example Let P be the lattice of all finite subsets of $\mathbf{N}$ ordered by inclusion. Then P is a distributive lattice and $B(n) = n!$. We will denote this poset as B.

Definition For $i, n \in \mathbf{N}$ let $\left[\begin{array}{c} n \\ i \end{array} \right]$ denote the number of elements z of rank i in an n-interval $[x, y]$. Note that since $B(i)B(n-i)$ maximal chains of $[x, y]$ pass through a given z of rank i, we have

$$\left[\begin{array}{c} n \\ i \end{array} \right] = \frac{B(n)}{B(i)B(n-i)} ,$$

so $\left[\begin{array}{c} n \\ i \end{array} \right]$ depends only on n and i. When $P =$ B then $B(n) = n!$ and $\left[\begin{array}{c} n \\ i \end{array} \right] = \binom{n}{i}$. The analogy with factorials is strengthened further by observing that $B(n) = A(n)A(n-1) \cdots A(1)$, where $A(i) = \left[\begin{array}{c} i \\ 1 \end{array} \right]$, the number of atoms in an i-interval.

Theorem *Let* P *be a binomial poset with factorial function* $B(n)$ *and incidence algebra* $I(P)$. *Define* $R(P) = \{f \in I(P) : f(x, y) = f(x', y')$ *if* $l(x, y) = l(x', y')\}$. *If* $f \in R(P)$ *then write* $f(n)$ *for* $f(x, y)$ *when* $l(x, y) = n$. *Then* $R(P)$ *is a subalgebra of* $I(P)$, *and we have an algebra isomorphism* $\phi : R(P) \to \mathbf{C}[[x]]$ *given by* $\phi(f) = \sum_{n \geq 0} f(n) x^n / B(n)$.

Proof Key point:

$$fg(x, y) = \sum_{z \in [x, y]} f(x, z) g(z, y) = \sum_{i=0}^{n} \left[\begin{array}{c} n \\ i \end{array} \right] f(i) g(n-i)$$

$\surd$

Proposition *Suppose* f^{-1} *exists in* $I(P)$ *(recall this is equivalent to* $f(x, x) \neq 0$ *for all* $x \in P$*). Then* $f^{-1} \in R(P)$.

Example Let $Z_n(\lambda)$ denothe the zeta polynomial in the variable λ of an n-interval $[x, y]$ of P. Then since $Z_n(\lambda) = \zeta^\lambda(x, y)$, we have

$$\sum_{n\geq 0} Z_n(\lambda) x^n / B(n) = \left(\sum_{n\geq 0} x^n / B(n) \right)^\lambda$$

Rational Generating Functions

Theorem: Rational Power Series in One Variable

Theorem *Let $\alpha_1, \alpha_2, \ldots, \alpha_d$ be a fixed sequence of complex numbers, $d \geq 1$ and $\alpha_d \neq 0$. The following conditions on a function $f : \mathbf{N} \to \mathbf{C}$ are equivalent:*

i. $\sum_{n\geq 0} f(n) x^n = P(x)/Q(x)$, where $Q(x) = 1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_d x^d$ and $P(x)$ is a polynomial in x of degree less than d.

ii. For all $n \geq 0$,

$$f(n + d) + \alpha_1 f(n + d - 1) + \alpha_2 f(n + d - 2) + \cdots + \alpha_d f(n) = 0\,.$$

iii. For all $n \geq 0$,

$$f(n) = \sum_{i=1}^{k} P_i(n) \gamma_i^n\,,$$

where $1 + \alpha_1 x + \alpha_2 x^2 + \cdots + \alpha_d x^d = \prod_{i=1}^{k} (1 - \gamma_i x)^{d_i}$, the γ_i's are distinct, and $P_i(n)$ is a polynomial in n of degree less than d_i.

Corollary *Suppose $f : \mathbf{N} \to \mathbf{C}$ satisfies any (or all) of the three equivalent conditions of the previous theorem, and preserve the notation of that theorem. The following conditions are equivalent:*

i. $P(x)$ and $Q(x)$ are relatively prime.

ii. The homogeneous linear recurrence with constant coefficients of least degree is the one given.

iii. $\deg P_i(n) = d_i - 1$ for $1 \leq i \leq k$.

Now consider rational functions without the restriction $\deg P < \deg Q$.

Proposition *Let $f : \mathbf{N} \to \mathbf{C}$ and suppose that $\sum_{n\geq 0} f(n) x^n = P(x)/Q(x)$. Then there is a unique finite set $E_f \subset \mathbf{N}$ (called the* exceptional set *of f) and a unique function $f_1 : E_f \to \mathbf{C}^*$ such that the function $g : \mathbf{N} \to \mathbf{C}$ defined by*

$$g(n) = \begin{cases} f(n), & \text{if } n \notin E_f \\ f(n) + f_1(n), & \text{if } n \in E_f, \end{cases}$$

satisfies $\sum_{n\geq 0} g(n) x^n = R(x)/Q(x)$ where R is a polynomial of degree less than Q. Moreover, assuming $E_f \neq \emptyset$ (i.e., $\deg P \geq \deg Q$), define $m(f) = \max\{i, i \in E_f\}$. Then:

i. $m(f) = \deg P - \deg Q$

ii. $m(f)$ is the largest integer n for which the recurrence fails to hold.

iii. Writing $Q(x) = \prod_1^k (1 - \gamma_i x)^{d_i}$ as in the theorem above, then $m(f)$ is the largest integer n for which it fails.

Proof Division. $\surd$

Proposition *Suppose $f : \mathbf{Z} \to \mathbf{C}$ satisfies*

$$f(n+d) + \alpha_1 f(n+d-1) + \cdots + \alpha_d f(n) = 0$$

for all $n \in \mathbf{Z}$. Thus $\sum_{n \geq 0} f(n)x^n = F(x)$ is a rational function. We then have

$$\sum_{n \geq 1} f(-n)x^n = -F\left(\frac{1}{x}\right)$$

as rational functions.

Corollary: Polynomials

Corollary *Let $f : \mathbf{N} \to \mathbf{C}$, and let $d \in \mathbf{N}$. The following three conditions are equivalent:*

i.

$$\sum_{n \geq 0} f(n)x^n = \frac{P(x)}{(1-x)^{d+1}},$$

where $P(x)$ is a polynomial with degree at most d.

ii. For all $n \geq 0$,

$$\sum_{i=0}^{d+1} (-1)^{d+1-i} \binom{d+1}{i} f(n+i) = 0\,.$$

In other words, $\Delta^{d+1} f(n) = 0$.

iii. $f(n)$ is a polynomial function of n of degree at most d. (Moreover, $f(n)$ has degree exactly d if and only if $P(1) \neq 0$.)

Linear Homogeneous Diophantine Equations

See Thesis.