215B Study Guide

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November 22, 1999

The Higher Homotopy Groups

(Almost verbatim from Hatcher)

How does $\pi_n(X, x_0)$ depend on the choice of the basepoint x_0? Given a path $\gamma : I \to X$ from $x_0 = \gamma(0)$ to another basepoint $x_1 = \gamma(1)$, we may associate to each map $f : (I^n, \partial I^n) \to (X, x_1)$ a new map $\gamma f : (I^n, \partial I^n) \to (X, x_0)$ obtained by shrinking the domain of f to a smaller concentric cube in I^n, and then inserting the path γ on each radial segment in the shell between this smaller cube and ∂I^n.

It is clear that a homotopy of γ or f through maps fixing ∂I or ∂I^n respectively, yields a homotopy of γf through maps $(I^n, \partial I^n) \to (X, x_0)$. Further basic properties are:

(1) $\gamma(f + g) \simeq \gamma f + \gamma g$. First deform f and g to be constant on the right and left halves of I^n repsecitvely, then we excise a progressively wider symmetri middle slab of $\gamma f + \gamma g$ until it becomes $\gamma(f + g)$.

(2) $(\gamma\eta)f \simeq \gamma(\eta f)$. Obvious.

(3) $1f \simeq f$ where 1 denotes the constant path. Obvious.

If we define $\beta_\gamma : \pi_n(X, x_0) \to \pi_n(X, x_1)$ by $\beta_\gamma([f]) = [\gamma f]$, then (1) shows that β_γ is a homomorphism, while (2) and (3) imply that β_γ is an isomorphism with inverse $\beta_{\bar{\gamma}}$ where $ga\bar{m}ma$ is the inverse path of γ, $\bar{\gamma}(s) = \gamma(1 - s)$. This if X is path-connected, different choices of basepoint x_0 yield isomorphic groups $\pi_n(X, x_0)$, which may then be written simply as $\pi_n(X)$.

Consider now loops γ at the basepoint x_0. Since $\beta_{\gamma\eta} = \beta_\gamma\beta_\eta$, the association $[\gamma] \mapsto \beta_\gamma$ defines a homomorphism from $\pi_1(X, x_0)$ to $Aut(\pi_n(X, x_0))$, the group of automorphisms of $\pi_n(X, x_0)$. This is called the action of π_1 on π_n. When $n = 1$ this is the action of π_1 on itself by inner automorphisms. When $n > 1$ this makes the abelian group $\pi_n(X, x_0)$ into a module over the group ring $\mathbf{Z}[\pi_1(X, x_0)]$. Elements of $\mathbf{Z}[\pi_1]$ are finite sums $\sum_i n_i\gamma_i$ with $n_i \in \mathbf{Z}$ and $\gamma_i \in \pi_1$, multiplication being defined by distributivity and the multiplication in π_1. The module structure on π_n is given by $\left(\sum_i n_i\gamma_i\right)\alpha = \sum_i n_i(\gamma_i\alpha)$ for $\alpha \in \pi_n$. For brevity, one sometimes says π_n is a π_1-module.

Definition A space with trivial π_1 action on π_n is called *n-simple*. *Simple* means n-simple for all n.

Observe π_n is a functor. Namely, a map $\phi : (X, x_0) \to (Y, y_0)$ induces $\phi_* : \pi_n(X, x_0) \to \pi_n(Y, y_0)$ defined by $\gamma_*([f]) = [\gamma f]$. γ_* is a well-defined and a homomorphism for $n \geq 1$. The functor properties $(\phi\psi)_* = \phi_*\psi_*$ and $1_* = 1$ are evident, as is the fact that if $\phi_t : (X, x_0) \to (Y, y_0)$ is a homotopy then $\phi_{0*} = \phi_{1*}$.

In particular, a homotopy equivalence $(X, x_0) \simeq (Y, y_0)$ in the basepointed sense induces isomorphisms on all homotopy groups π_n. This is true even if basepoints are not required to be stationary during homotopies (exercise).

Proposition *If $p : (\tilde{X}, \tilde{x}_0) \to (X, x_0)$ is a covering space, then $p_* : \pi_n(\tilde{X}, \tilde{x}_0) \to \pi_n(X, x_0)$ is an isomorphism for all $n \geq 2$.*

Proof If $n \geq 2$ the map lifting criterion implies that a map $(S^n, s_0) \to (X, x_0)$ lifts to $(\tilde{X}, \tilde{x}_0)$ since $\pi_1(S^n) = 0$. So p_* is surjective on π_n. Injectivity is immediate from the covering homotopy property. $\surd$

In particular, $\pi_n(X, x_0) = 0$ for $n \geq 2$ whenever X has a contractible universal cover. Example: S^1, or $T^2 = S^1 \times S^1$, or T^n (compare with homology groups $H_i(T^n) = \mathbf{Z}^{C(n,i)}$). Spaces with $\pi_n = 0$ for $n \geq 2$ are sometimes called *aspherical*.

Proposition *For a product $\prod_\alpha X_\alpha$ of an arbitrary collection of path-connected spaces X_α, there are isomorphisms $\pi_n(\prod_\alpha X_\alpha) \approx \prod_\alpha \pi_n(X_\alpha)$ for all n.*

The Long Exact Sequence

Proof A map $f : Y \to \prod_\alpha X_\alpha$ is the same thing as a collection of maps $f_\alpha : Y \to X_\alpha$. Taking Y to be S^n and $S^n \times I$ gives the result. $\surd$

Definition To define relative groups $\pi_n(X, A, x_0)$ for a pair (X, A) with basepoint $x_0 \in A$, regard I^{n-1} as the face of I^n with the last coordinate $s_n = 0$ and let J^{n-1} be the closure of $\partial I^n - I^{n-1}$, the union of the remaining faces of I^n. Then $\pi_n(X, A, x_0)$ for $n \geq 1$ is defined to be the set of homotopy classes of maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, with homotopies through maps of the same form. $\pi_0(X, A, x_0)$ is undefined. Note $\pi_n(X, x_0, x_0) = \pi_n(X, x_0)$.

A sum operation is defined in $\pi_n(X, A, x_0)$ by the same formulas as for $\pi_n(X, x_0)$ execpt that the coordinate s_n now plays a special role and is no longer available for the sum operation. Thus $\pi_n(X, A, x_0)$ is a group for $n \geq 2$, and the group is abelian for $n \geq 3$. For $n = 1$, $\pi_1(X, A, x_0)$ is the set of homotopy classes of pahts in X from a varying point in A to the fixed basepoint $x_0 \in A$. In general this is not a group in any natural way.

Just as $\pi_n(X, x_0)$ can be regarded as homotopy classes of maps $(S^n, s_0) \to (X, x_0)$, so $\pi_n(X, A, x_0)$ can be viewed as homotopy classes of maps $(D^n, S^{n-1}, s_0) \to (X, A, x_0)$.

A map $\phi : (X, A, x_0) \to (Y, B, y_0)$ induces maps $\phi_* : \pi_n(X, A, x_0) \to \pi_n(Y, B, y_0)$ which are homomorphisms for $n \geq 2$ and have properties analogous to those in the absolute case: $(\phi\psi)_* = \phi_*\psi_*$, $1_* = 1$, and $\phi_* = \psi_*$ if $\phi \simeq \psi$.

The relative groups $\pi_n(X, A, x_0)$ fit into a long exact sequence

$$\cdots \to \pi_n(A, x_0) \xrightarrow{i} \pi_n(X, x_0) \xrightarrow{j} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, x_0) \to \cdots \to \pi_0(X, x_0)$$

Here i is induced by inclusion, j comes fromregarding a map $(I^n, \partial I^n) \to (X, x_0)$ as a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, and ∂ from restring maps $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1}. The maps i, j, and ∂ are clearly homomorphisms when group structures are defined in domain and range.

Theorem *This sequence is exact.*

The proof will use a general observation: A map $f : (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ reperesents zero in $\pi_n(X, A, x_0)$ iff it is homotopic rel ∂I^n to a map with image contained in A. For if we have such a homotopy to a map g, then $[f] = [g]$ in $\pi_n(X, A, x_0)$, and $[g] = 0$ via the homotopy obtained by composing g with a deformation retraction of I^n onto J^{n-1}. Conversely, if $[f]=0$ via a homotopy $F: I^{n+1} \to X$, then by restricting F to a family of n-disks in I^{n+1} starting with I^n and ending with J^n, all these disks having the same boundary, then we get a homotopy from f to a map into A, stationary on ∂I^n.

Proof With just about the same effort we can derive the long exact sequence of a triple (X, A, B, x_0) with $x_0 \in B \subset A \subset X$:

$$\cdots \to \pi_n(A, B, x_0) \xrightarrow{i} \pi_n(X, B, x_0) \xrightarrow{j} \pi_n(X, A, x_0) \xrightarrow{\partial} \pi_{n-1}(A, B, x_0) \to \cdots \to \pi_0(X, A, x_0)$$

When $B = x_0$ this reduces to the exact sequence for the pair (X, A, x_0), though the later sequence continues on two more steps to $\pi_0(X, x_0)$.

Exactness at $\pi_n(X, B, x_0)$: The composition ji is zero since a map $(I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ represents zero in $\pi_n(X, A, x_0)$ by the remarks preceding the proof. Conversely, if $f: (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ represents zero in $\pi_n(X, A, x_0)$, then it is homotopic rel ∂I^n to a map with image in A, hence the class $[f] \in \pi_n(X, B, x_0)$ is in the image of i.

Exactness at $\pi_n(X, A, x_0)$: The composition ∂j is zero since the restriction of a map

$$(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$$

to I^{n-1} has image lying in B, hence represents zero in $\pi_{n-1}(A, B, x_0)$. Conversely, suppose the restriction of $f: (I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$ to I^{n-1} represents zero in $\pi_{n-1}(A, B, x_0)$, so there is a homotopy F from $f|I^{n-1}$ to a map with image in B, rel ∂I^{n-1}. We can tack F onto f to get a new map $f_1 : (I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$ which, as a map $(I^n, \partial I^n, J^{n-1}) \to (X, A, x_0)$, is homotopic to f by the homotopy which tacks on increasingly longer initial segments of F. So $[f] = [f_1] \in Im(j)$.

Exactness at $\pi_n(A, B, x_0)$: The composition $i\partial$ is zero since the restriction of a map $f: (I^{n+1}, \partial I^{n+1}, J^n) \to (X, B, x_0)$ to I^n is homotopic rel ∂I^n to a constant map via f itself. The converse is trivial if B is a point, since a nullhomotopy $f_t : (I^n, \partial I^n) \to (X, x_0)$ of $f_0 : (I^n, \partial I^n) \to (A, x_0)$ gives a map $F: (I^{n+1}, \partial I^{n+1}, J^n) \to (X, A, x_0)$ with $\partial([F]) = [f_0]$. Thus the proof is finished in this case. For a general B, let F be a nullhomotopy of $f: (I^n, \partial I^n, J^{n-1}) \to (A, B, x_0)$ trhough maps $(I^n, \partial I^n, J^{n-1}) \to (X, B, x_0)$, and let g be the restriction of F to $I^{n-1} \times I$. Reparametrizing the n^{th} and $(n+1)^{\text{st}}$ coordinates so that instead of g being on the left of $I \times I$ and f at the bottom, g followed by f is at the bottom and x_0 is everywhere else. But as we noted in the preceding paragraph, tacking g onto f gives the same element of $\pi_n(A, B, x_0)$. $\surd$

Example Let CX be the cone on a path-connected space X, so CX is the quotient space of $X \times I$ obtained by collapsing $X \times \{0\}$ to a point. We can view X as the subspace $X \times \{1\} \subset CX$. Since CX is contractible, the long exact sequence of homotopy groups gives isomorphisms $\pi_n(CX, X, x_0) \approx \pi_{n-1}(X, x_0)$. Taking $n = 2$, we can thus realize any group G, abelian or not, as a relative π_2 by choosing X to have $\pi_1(X) \approx G$.

The long exact sequence of homotopy groups is clearly natural: A map $(X, A, x_0) \to (Y, B, y_0)$ induces a map between the long exact sequences for these two pairs of spaces, with commuting squares.

For relative homotopy groups there are basepoint-chainge isomorphism β_γ analogous to those in the absolute case. One starts with a path γ in $A \subset X$ from x_0 to x_1, and this induces $\beta_\gamma : \pi_n(X, A, x_1) \to \pi_n(X, A, x_0)$, $\beta_\gamma[f] = [\gamma f]$, where γf is defined as f in a smaller rectangle resting on one side, with lines connecting this rectangle (which has boundary at x_1) via paths γ to the boundary of the domain. The basic properties of these relative β_γ's parallel closely those developed earlier in the absolute case. Note however that, even though absolute homotopy groups $\pi_n(X, x_0)$ are the special case of relative groups $\pi_n(X, A, x_0)$ in which $A = x_0$, the relative β_γ's do not specialize to the absolute β_γ's.

Restricting γ to loops at the basepoint, the association $\gamma \mapsto \beta_\gamma$ defines an action of $\pi_1(A, x_0)$ on $\pi_n(X, A, x_0)$, as in the absolute case. In fact, it is clear from the definitions that $\pi_1(A, x_0)$ acts on the whole long exact sequence of homotopy groups for (X, A, x_0), the action commuting with the various maps in the sequence.

Whitehead's Theorem

Whitehead's Theorem *If a map $f : X \to Y$ between connected CW complexes induces isomorphisms $f_* : \pi_n(X) \to \pi_n(Y)$ for all n, then f is the homotopy equivalence. In case f is the inclusion of a subcomplex $X \hookrightarrow Y$, the conclusion is tronger: X is a deformation retract of Y.*

Lemma *Let (X, A) be a CW pair and (Y, B) a pair with Y path-connected and B nonempty. Suppose that for each $n > 0$ such that $X - A$ has cells of dimension n, $\pi)n(Y, B, y_0)$ for all $y_0 \in B$. Then every map $f : (X, A) \to (Y, B)$ is homotopic* rel A *to a map $X \to B$.*

Proof (Lemma) Assume inductively that f has already been homotoped to take the skeleton X^{k-1} to B. If Φ is the characteristic map of a cell e^k of $X - A$, the composition $f\Phi : (D^k, \partial D^k) \to (Y, B)$ can be homotoped into B rel ∂D^k since $\pi_k(Y, B) = 0$ by hypothesis. This homotopy induces a homotopy of f on the quotient space $X^{k-1} \cup e^k$ of $X^{k-1} \sqcup D^k$, a homotopy rel X^{k-1}. Doing this for all k-cells of $X - A$ simultaneously, and taking the constant homotopy on A, we obtain a homotopy of $f|X^k \cup A$ to a map into B. By the homotopy extension property, this homotopy extends to a homotopy defined on all of X, completing the induction step. The induction starts with the case $k = 0$, where we use the assumption that Y is path-connected and B is nonempty to deform f to take 0-cells into B.

If the cells of $X - A$ are of bounded dimension, finitiely many applications of the induction step finish the proof. To handle the case that $X - A$ has cells of arbitrarily large dimension, we perform the homotopy of the inductions step during the t-interval $[1 - 1/2^k, 1 - 1/2^{k+1}]$. Any finite skeleton X^k is eventually stationary under these homotopies, hence we have a well-defined homotopy f_t, $t \in [0, 1]$, with $f_1(X) \subset B$. $\surd$

Aside CW complexes with cells in arbitrarily large dimension are given the weak topology: A set $A \subset X$ is open (or closed) iff $A \cap X^n$ is open (or closed) in X^n for each n.

Proof (Whitehead's Theorem) In the special case that f is the inclusion of a subcomplex, consider the long exact sequence of homotopy groups for the pair (Y, X). Since f induces isomorphisms on all homotopy groups, the relative groups $\pi_n(Y, X)$ are zero, so (Y, X) is ∞-connected ($\pi_i(X, A, x_0) = 0$ for all $x_0 \in A$ and all i). Applying the lemma to the identity map $(Y, X) \to (Y, X)$ then yields a deformation retraction of Y onto X.

The general case can be reduced to this special case by using mapping cylinders. Recall that the mapping cylinder M_f of a map $f : X \to Y$ is the quotient space of the disjoint union of $X \times I$ and Y under the identification $(x, 1) \sim f(x)$. Thus M_f contains both $X = X \times \{0\}$ and Y as subspaces, and M_f deformation retracts onto Y. The map f becomes the composition of the inclusion $X \hookrightarrow M_f$ with the retraction $M_f \to Y$ which is a homotopy equivalence, so it suffices to show the inclusion $X \hookrightarrow M_f$ is a homotopy equivalence if it induces isomorphism on homotopy groups, i.e., if the relative groups $\pi_n(M_f, X)$ are all zero. The easiest way to complete the argument now is to appeal to cellular approximation (below) which says f can be homotoped to a cellular map, and then M_f will be a CW complex with X as a subcomplex, so the special case already proved applies.

Alternatively, the following argument for deformation retracting M_f to X can be used instead. First apply the preceding lemma to homotope the inclusion $(X \cup Y, X) \hookrightarrow (M_f, X)$ to a map into X, staying fixed on X. Since $(M_f, X \cup Y)$ satisfies the homotopy extension property, this homotopy extends to a homotopy from the identity map of M_f to a map $g : M_f \to M_f$ taking $X \cup Y$ into X. Then apply the lemma again to the composition $(X \times I \sqcup Y, X \times \partial I \sqcup Y) \to (M_f, X) \xrightarrow{g} (M_f, X)$ to finish the construction of a deformation retraction of M_f onto X. $\surd$

Whitehead's theorem does not say that CW complexes X and Y with isomorphic homotopy groups are homotopy equivalent, since one needs a map $X \to Y$ inducing the isomorphisms $\pi_n(X) \approx \pi_n(Y)$. For example, consider $X = \mathbf{R}P^2$ and $Y = S^2 \times \mathbf{R}P^\infty$. These both have fundamental group $\mathbf{Z}_2$, and the earlier proposition says covers agree with their bases on higher homotopy groups, so since their universal covers S^2 and $S^2 \times S^\infty$ are homotopy equivalent (S^∞ being contractible), X and Y have isomorphic higher homotopy groups. But $\mathbf{R}P^2$ and $S^2 \times \mathbf{R}P^\infty$ are not homotopy equivalent since their homology groups are vastly different, $S^2 \times \mathbf{R}P^\infty$ having nonvanishing homology in infinitely many dimensions since it retracts onto $\mathbf{R}P^\infty$. Another paif of non-homotopy-equivalent CW complexes with isomorphic homotopy groups is S^2 and $S^3 \times \mathbf{C}P^\infty$, as we shall see later.

Cellular Approximation

Definition A map $f : X \to Y$ between CW complexes is *cellular* if $f(X^n) \subset Y^n$ for all n.

Cellular Approximation Theorem *Every map $f : X \to Y$ of CW complexes is homotopic to a cellular map. If f is already cellular on a subcomplex $A \subset X$, the homotopy may be taken to be stationary on A.*

It follows that maps of CW pairs $f : (X, A) \to (Y, B)$ can also be deformed to be cellular. One can first deform $f : A \to B$ to be cellular, then extend this to a homotopy of f on all of X, then deform the resulting map to be cellular staying fixed on A. This produces a homotopy $f_t : (X, A) \to (Y, B)$ from f to a cellular map.

Corollary $\pi_i(S^n) = 0$ *for* $i < n$.

Lemma *Let $f : I^n \to Z$ be a map, where Z is obtained from a subspace W by attaching a cell e^m. Then there is a homotopy f_t of f, stationary on $f^{-1}(W)$, to a map f_1 such that there is a simplex $\Delta^m \subset e^m$ with $f_1^{-1}(\Delta^m)$ a union of finitely many convex polyhedra, on each of which f_1 is the restriction of a surjective linear map $\mathbf{R}^n \to \mathbf{R}^m$.*

Definition A space with basepoint x_0 is called *n-connected* if $\pi_i(X, x_0) = 0$ for $i \leq n$. Thus 0-connected means path-connected, and 1-connected is the same as simply-connected. Similarly, in the relative case, a pair (X, A) is called *n-connected* if $\pi_i(X, A, x_0) = 0$ for all $i \leq n$. ∞-connected if holds for all i.

Corollary *A CW pair (X, A) is n-connected if all cells of $X - A$ have dimension greater than n. Hence the inclusion $A \hookrightarrow X$ induces isomorphisms on π_i for $i < n$ and a surjection on π_n.*

Cellular Models

Definition A map $f : X \to Y$ is called a *weak homotopy equivalence* if it induces isomorphisms $\pi_n(X, x_0) \to \pi_n(Y, f(x_0))$ for all n and all choices of basepoint x_0.

Whitehead's theorem can be restated as saying that a weak homotopy equivalence between CW complexesis a homotopy equivalence. But in general a weak homotopy equivalence is strictly weaker than homotopy equivalence. For example the "quasi-circle," formed by $\sin(1/x)$ for $x \in (0, c)$ union $0 \times [-1, 2]$ union a path from $(0, 2)$ to $(c, \sin(1/c))$, is a noncontractible space whose homotopy groups are all trivial.

Proposition *A weak homotopy equivalence $f : X \to Y$ induces isomorphisms $f_* : H_n(X; G) \to H_n(Y; G)$ and $f^* : H^n(Y; G) \to H^n(X; G)$ for all n and all coefficient groups G.*

Excision for Homotopy Groups

The Hurewicz Theorem

Hurewicz Theorem *There is a (obvious map) $\pi_n(X, x_0) \to H_nX$ got by $(f : S^n, * \mapsto X, x_0) \mapsto f_*[S^n] \in H_n$ (where $[S^n]$ is a fundamental class: a generator of $H_nS^n \cong \mathbf{Z}$).*

Alternatively think of f as being sub-divided as a singular chain, cf. case $n = 1$: a loop *is* a singular 1-cycle.

Lemma *H is a homomorphism.*

Proof (Lemma) Obviously well-defined, since $f \simeq g \implies f_* = g_*$. Also, if $p : S^n \to S^n \vee S^n$ is pincing the equator, then

$$(f + g)_*[S] = [(f \vee g) \circ p]_*[S] = (f \vee g)_*(i_*[S] + j_*[S]) = f_*[S] + g_*[S]$$

$\surd$

Theorem $\pi_nS^n \cong \mathbf{Z}$. *Reminder: $\pi_iS^n = 0$ for $i < n$ by cellular approximation.*

Proof $\deg : \pi_nS^n \to \mathbf{Z}$ is a well-defined surjective homomorphism [recall: $f_*[S] = d[S]$ in $H_nS^n \cong \mathbf{Z}$ (homomorphism by proof of previous lemma and surjective by suspending maps $z \mapsto z^d$). So, we need to show it's injective. Suppose $f : S^n \to S^n$ has degree zero. Recall the local computation: if there exists a *regular value* $p \in S^n$ whose preimage $f^{-1}(p) = \{x_1, \ldots, x_k\}$, each with a neighborhood U_i mapping homeomorphically to a neighborhood U of p, then $\deg f = \sum_{i=1}^{k}(\deg f : H_n(U_i, U_i - x_i) \to H_n(U, U - p))$ (the sum of the "local degrees"). Recall $H_n(U, U - x_i) \cong H_{n-1}(S^{n-1})$ (by l.e.s.; $U - x_i \cong \mathbf{R}^n - x_i \simeq S^{n-1}$). Claim: we can always produce a regular value by saying f is homotopic to a smooth map (with an arbitrarily small homotopy; Sard's theorem $\implies$ regular values are dense in S^n).

Remark Counting preimages formula is just obtained by taking a bump function form as a generator of H^nS^n.

Alternatively, we could use a simplicial approximation.

Simplicial Approximation Theorem *Any continuous map $f : |K| \to |L|$ is homotopic to a simplicial map (carries simplexes to simplexes affinely) from $(K)^{(r)} \to L$ where $(K)^{(r)}$ is the r^{th} barycentric subdivision of K for (large) r.*

Take a simplicial representative for a homotopy class. The center of any n-simplex is a regular value, so $f^{-1}(p) = \{x_1, \ldots, x_n\}$ have required neighborhoods. Assume f has degree 0 so $(f|_{U_i})_* : H_n(U, U - x_i)$ local degrees cancel in pairs. Construct a homotopy from f to a map with 2 fewer preimages of p. Continuing this will homotop f to a map $S^n \to S^n - \{p\}$ which is null-homotopic. Connect a $\{+,-\}$ pair with an arc in $S^n \times I$ (if $n = 1$, make them adjacent!). Define the homotopy in sections: take a neighborhood $B^n \times I$ of the arc on its boundary $S^{n-1} \times I$, the restriction of f to the ends have *same* degree. By induction (on n, assuming $\pi_{n-1}S^{n-1} \cong \mathbf{Z}$), these are then homotopic; fill in this homotopy $S^{n-1} \times I \to S^{n-1}$ on the boundary of the tube. Now extend inside the tube: we have a map $\partial(\text{tube} = S^n) = (\text{disc}) \cup (S^{n-1} \times I) \cup \text{disc} \to$ nbhd of p (=ball). So since the target ball is contractible, the map is null homotopic and we fill in the inside of the tube using the null-homotopy ($\cong B^{n+1}$). Pick neighborhoods around other preimages and "fix them" during the homotopy. On the boundary of the remaining region, we have a map to $S^n - \{p\}$ which can therefore be extended to fill in the region in the same way. $\surd$

Corollary $\pi_n(\bigvee_\alpha S^n) \cong \bigoplus_\alpha \pi_n S^n \cong \oplus_\alpha \mathbf{Z}$ *when* $n > 1$.

Proof We know $H_*(X \vee Y) = H_*X \oplus H_*Y$ and $\pi_i(X \times Y) = \pi_i X \times \pi_i Y$. Look at $X \vee Y$ as $X \vee Y = X \times \{y_0\} \cup \{x_0\} \times Y \hookrightarrow X \times Y$. Use l.e.s. for $\bigvee_\alpha S^n \hookrightarrow \prod_\alpha S^n$. Since $\prod S^n$ is obtained from $\bigvee S^n$ by attaching cells of dim $\geq 2n$, $\pi_i(\prod S^n, \bigvee S^n) = 0$ for $i < 2n$ (by CAT). At least for $i = n > 1$, this implies $\pi_n(V) \cong \pi_n(\prod S^n) = \bigoplus Z$. $\surd$

Relative Hurewicz *For CW pair* (X, A)*,*

(1) $\pi_n(X, A) \to H_r(X, A)$ *is a homomorphism* $(n \geq 2)$*.*

(2) If $\pi_1(A) = 0$ *and* $\pi_i(X, A) = 0 \forall i < n$ $(n \geq 2)$ *then* $H_i(X, A) = 0$ *and first nonzero are isomorphic.*

(3) If $\pi_1(A) \neq 0$ *then* $\pi_i(X, A)$ *are all* $\mathbf{Z}\pi_1 A$ *modules* $(i \geq 3)$. $\pi_2(X, A)$ *(not necessarily abelian) is a* $\pi_1 A$*-group (there is a* π_1 *action).*

Observe $[f]^\gamma$ and $[f]$ map to same elments in $H_i(X, A)$ under Hurewicz.

Relative Hurewicz Theorem *First non-vanishing groups satisfy*

$$\frac{\pi_n(X, A)}{\langle f^\gamma - f \rangle} \cong H_n(X, A)$$

Angle brackets mean normal subgroup generated by $f \in \pi_n, \gamma \in \pi_1$*, and* $-$ *indicates muliplication in the group* π_n*.*

Corollary *Alternative version of Whitehead's Theorem: If* $f : X \to Y$ *induces an isomorphism on* π_1 *and its lift to universal covers induces isomorphisms on Homology, then* X *and* Y *are homotopic.*

Eilenberg-Maclane Spaces

Fibrations

Fiber Bundles

Definition A *Fiber bundle* is a map $\pi : E \to B$ together with a space F (*fiber*) and the condition of *local triviality*: every point $x \in B$ has a neighborhood U such that $\pi^{-1}(U) \cong U \times F$ and the homeomorphism is *fiber-preserving.*

Example Hopf fibration: (fiber is S^1, S^3 is the total space, and S^2 is the base.)

$$\begin{array}{ccc} S^1 & \to & S^3 \\ & & \downarrow \pi \\ & & S^2 \end{array}$$

defined by $(z,w) \sim \lambda(z,w)$ for $\lambda = e^{i\theta}$, $(z,w) \in S^3 \subset \mathbf{C}^2$. The quotient is $(\mathbf{C}^2 - \{0\})/\mathbf{C}^* = \mathbf{C}P^1$, the Riemann sphere, or S^2. To see this is locally trivial, it's easier to say $S^3 \to S^2$ by $(z,w) \mapsto z/w \in \mathbf{C}P^1$. Then the open sets $(\mathbf{C}P^1 - \{0\}), (\mathbf{C}P^1 - \{\infty\})$ do the trick.

Example Two others corresponding to quaternions ($\mathbf{H}$) and octonians.

$$\begin{array}{ccccc} \mathbf{H} \supseteq S^3 & \to & S^7 \subseteq \mathbf{H}^2 & S^7 \to & S^{15} \\ & & \downarrow & & \downarrow \\ & & S^4 & & S^8 \end{array}$$

Definition A *Lie group* G is a smooth manifold which is a topological group with smooth maps m, i. Eg. orthogonal group of matrices.

If H is a closed subgroup of a Lie group, then

$$\begin{array}{ccc} H & \to & G \\ & & \downarrow \\ & & G/H \end{array}$$

is a fibration.

Definition A *fibration* (or *Hurewicz* fibration) is a map $E \xrightarrow{\pi} B$ which satisfies the Homotopy Lifting Property for all spaces X, i.e. given a homotopy $X \times I \xrightarrow{H} B$ and a lift $\tilde{f}$ of $f = H|_0$ there exists a lift $\tilde{H} : X \times I \to E$ with $\tilde{H}|_0 = \tilde{f}$.

Remark A *Serre* fibration satisfies the HLP for CW-complexes.

Remark Fiber bundles are fibrations.

If $E \xrightarrow{\pi} B$ is a fibration $b_0 \in B$, $F = \pi^{-1}(b_0)$, $e_0 \in F \subseteq E$, then there is a l.e.s.

$$\cdots \xrightarrow{\partial_*} \pi_n(F, e_0) \xrightarrow{i_*} \pi_n(E, e_0) \xrightarrow{\pi_*} \pi_n(B, b_0) \xrightarrow{\partial_*} \pi_{n-1}(F, e_0) \to \cdots$$
$$\cdots \to \pi_0 F \to \pi_0 E \to \pi_0 B$$

Proof This is really the l.e.s. of relative homotopy for the pair (E, F) once we prove that $\pi_* : \pi_n(E, F, e_0) \to \pi_n(B, b_0)$ is an isomorphism. $\surd$

Example Covering spaces: $\pi_n(E) \cong \pi_n(B) \forall n \geq 2$. For $n = 1$, p_* is injective $0 \to \pi_1 E \to \pi_1 B \to \pi_0 F \to 0$, therefore (fiber)$\leftrightarrow \pi_1 B / p_* \pi_1 E$.

Example Hopf fibration $\pi_3(S^2) \cong \pi_2(S^2) \cong \mathbf{Z}$ (last $\cong$ by Hurewicz).

$$\begin{array}{ccccccccc} S^1 & \to & S^\infty & \mathbf{Z}_2 & \to & S^\infty & S^3 & \to & S^\infty \\ & & \downarrow & & & \downarrow & & & \downarrow \\ & & \mathbf{C}P^\infty & & & \mathbf{R}P^\infty & & & \mathbf{H}P^\infty \end{array}$$

S^∞ is *contractible*: topology on $\mathbf{R}^\infty$ is a weak topology. It consists of finite-length sequences $\{(a_0, a_1, \ldots, a_n, 0, \ldots)\}$. In other words $\mathbf{R} \subseteq \mathbf{R}^2 \subseteq \mathbf{R}^3 \subseteq \cdots \subseteq \mathbf{R}^\infty$ characterized by $f : \mathbf{R}^\infty \to X$ continuous $\iff$ $f|_{\mathbf{R}^n}$ continuous for all n. S^∞ gets a CW complex structure with 2 cells in each dimension. Contractible by Whitehead: $\{*\} \hookrightarrow S^\infty$ induces a $\cong$ on homotopy groups.

So

$$\begin{array}{cc}
\pi_i(\mathbf{R}P^\infty) = \begin{cases} \mathbf{Z}_2 & i = 1 \\ 0 & \text{otherwise} \end{cases} & \pi_i(\mathbf{C}P^\infty) = \begin{cases} \mathbf{Z} & i = 2 \\ 0 & \text{otherwise} \end{cases} \\
\subseteq \mathbf{R}P^1 \subseteq \mathbf{R}P^2 \subseteq \cdots \text{ weak limit} & * \subseteq \mathbf{C}P^1 \subseteq \mathbf{C}P^2 \subseteq \cdots \\
K(\mathbf{Z}_2, 1) & K(\mathbf{Z}, 2)
\end{array}$$

($K(\pi, n)$ if only one non-zero homotopy group, "Eilenberg-Maclane" spaces)

Example $\pi_i(\Omega X) \cong \pi_{i+1}(X)$

$$\begin{array}{ccc}
\Omega_{x_0} X \to \mathcal{P}_{x_0} X = & \{\gamma : I \to X, \gamma(0) = x_0\} \simeq \{*\} \\
& \downarrow {\scriptstyle \pi : \gamma \mapsto \gamma(1)} \\
& X_{x_0}
\end{array}$$

Recall HLP:

$$\begin{array}{ccc}
x \times \{0\} & \xrightarrow{\tilde{H}_0} & E \\
\downarrow & \nearrow \tilde{H} & \downarrow \pi \\
X \times I & \xrightarrow{H} & B
\end{array}$$

This is a fibration if for all X, H, $\tilde{H}_0$ there exists a $\tilde{H}$ which makes this diagram commute.

Theorem *A fiber bundle has HLP for all CW complexes (we have uniqueness for covering spaces).*

Theorem *If $E \xrightarrow{\pi} B$ is a fibration with path-connected B, fibers at points b_0, b_1 are homotopy-equivalent.*

Theorem *Any map $f : X \to Y$ with Y path-connected can be replaced by a fibration $X \cong E_f \xrightarrow{\pi} Y$.*

Definition

$$E_f = \{(x, \gamma) | x \in X, \gamma \text{ is a path in } Y \text{ from } f(x)\} \subseteq X \times \mathcal{P}Y$$

Obstruction Theory

Also get homework problems relating to: Topological groups and H-spaces Smash product Loop space Ω and suspension Σ Mapping cone Whitehead product

And deal with: Universal Coefficient Theorem Cayley graph/complex (copy definitions)